Remarks on the vanishing viscosity problem with Dirichlet boundary conditions

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The Euler and Navier-Stokes equations

- Incompressible Navier-Stokes

$$\partial_t u^{NS} + u^{\mathrm{NS}} \cdot \nabla u^{\mathrm{NS}} - \nu \Delta u^{\mathrm{NS}} + \nabla p^{\mathrm{NS}} = 0, \quad \nabla \cdot u^{\mathrm{NS}} = 0,$$

- Incompressible Euler

$$\partial_t u^{\mathrm{E}} + u^{\mathrm{E}} \cdot \nabla u^{\mathrm{E}} + \nabla p^{\mathrm{E}} = 0, \quad \nabla \cdot u^{\mathrm{E}} = 0.$$

- Boundary conditions: Dirichlet for Navier-Stokes

$$u^{\mathrm{NS}}_{|\,\partial\Omega} = 0,$$

and non-penetrating for Euler

$$u^{\mathrm{E}}_{|\,\partial\Omega} \cdot n = 0.$$

The question of inviscid limit

- Initial conditions are asymptotically the same:

$$\|u_0^{NS} - u_0^{E}\|_{L^2} \to 0 \quad \text{as} \quad \nu \to 0.$$

- Finite time horizon: fix $T > 0$.
- For simplicity, fix: $d = 2$ and $\Omega = \mathbb{H}$.
- Navier-Stokes energy inequality:

$$\|u^{NS}(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u^{NS}(s)\|_{L^2}^2 ds \leq \|u_0^{NS}\|_{L^2}^2.$$

- Space of convergence: the energy space $L^\infty(0, T; L^2(\mathbb{H}))$.
- The problem:

$$\sup_{t\in[0,T]} \|u^{NS}(t) - u^{E}(t)\|_{L^2} \to 0 \quad \text{as} \quad \nu \to 0 \quad ???$$

- Smooth background Euler solution: $u_0^E \in H^s(\mathbb{H})$, for some $s > 2$.
- C_E is any constant that depends on $\|u^E\|_{L^\infty(0,T;H^s(\mathbb{H}))}$.

Kato ('84) and friends

- Kato ('84): the inviscid limit holds if and only if

$$\lim_{\nu\to 0} \nu \int_0^T \int_{x_2 \le O(\nu)} |\nabla u^{NS}(x,t)|^2 dxdt = 0$$

- Temam-Wang ('98), and Wang ('01):

only tangential gradients, but thicker layer $\delta(\nu)$: $\lim_{\nu\to 0} \frac{\delta(\nu)}{\nu} = 0$.

- Kelliher ('08): inviscid limit holds if and only if

$$\omega^{NS} \to \omega^{E} - u_1^{E} \mu_{\partial\mathbb{H}} \quad \text{in} \quad (H^1(\mathbb{H}))^*$$

- Bardos-Titi ('15): inviscid limit holds if and only if

$$\nu\omega^{NS} \rightharpoonup 0 \quad \text{in} \quad D'([0,T], \partial\mathbb{H})$$

Further "positive" results on the inviscid limit

- Masmoudi ('98): inviscid limit holds if $-\nu\Delta$ is replaced by anisotropic viscosity $-\nu_1\partial_{yy} - \nu_2\partial_{xx}$, with $\nu_1/\nu_2 \to 0$
- Lopes Filho-Mazzucato-Nussenzveig Lopes-Taylor ('08): vanishing viscosity limits holds for circularly symmetric 2D flows on a rotating boundary
- similar positive results in other symmetric geometries: Iftimie-Lopes Filho-Nussenzveig Lopes ('03), Lopes Filho-Kelliher-Nussenzveig Lopes ('09); Mazzucato, Taylor ('11);
- Guo-Nguyen ('15): inviscid limit holds for a steady moving plate
- Bardos-Szekelyhidi-Wiedemann ('14): weak-strong uniqueness if Hölder near the boundary
- Bardos-Nguyen ('14): Kato-type results for compressible fluids

Inviscid limit holds if the Prandtl expansion is valid

Theorems(!): if the initial datum obeys [...] then inviscid limit holds.

- Sammartino-Caflisch ('98): inviscid limit holds if the initial datum is real analytic in all variables.
- Maekawa ('14): inviscid limit holds if the initial vorticity is identically vanishing near $\partial\mathbb{H}$.

Asymptotic Expansions in the inviscid limit: Prandtl

- In the BL: $u_1^{\mathrm{NS}}|_{y=0}$ has to jump from 0 to $u_1^{\mathrm{E}}|_{y=0} = \mathcal{O}(1)$.
- In the BL: dominating viscous term $\nu \partial_{yy} u_1^{\mathrm{NS}} = \mathcal{O}(1)$, so that the thickness of the BL should be

$$\varepsilon = \sqrt{\nu}$$

- For $\nu \ll 1$, it is natural to consider the asymptotic expansion

$$u^{\mathrm{NS}} = u^{(\mathrm{NS},0)} + \varepsilon u^{(\mathrm{NS},1)} + \varepsilon^2 u^{(\mathrm{NS},2)} + \dots$$

  where as before $\varepsilon = \sqrt{\nu}$
- outside of the BL: $u^{(\mathrm{NS},0)} \approx u^{\mathrm{E}}$
- inside the BL: $u^{(\mathrm{NS},0)} \approx u^{\mathrm{P}}$
- let $Y = y/\varepsilon$ be the boundary layer variable
- Prandtl plugs in the ansatz:

$$u^{(\mathrm{NS},0)}(x, y) \approx (u_1^{\mathrm{P}}(x, Y), \varepsilon u_2^{\mathrm{P}}(x, Y))$$

  in the Navier-Stokes equations, and formally sends ε to 0

The Prandtl boundary layer equations

- In the limit we obtain the Prandtl boundary layer equations:

$$\begin{aligned}
&\partial_t u_1^{\mathrm{P}} - \partial_{YY} u_1^{\mathrm{P}} + u_1^{\mathrm{P}} \partial_x u_1^{\mathrm{P}} + u_2^{\mathrm{P}} \partial_Y u_1^{\mathrm{P}} + \partial_x p^{\mathrm{P}} = 0 \\
&\partial_Y p^{\mathrm{P}} = 0 \\
&\partial_x u_1^{\mathrm{P}} + \partial_Y u_2^{\mathrm{P}} = 0
\end{aligned}$$

- Boundary conditions

$$\begin{aligned}
\lim_{Y\to\infty} u_1^{\mathrm{P}} &= u_1^{\mathrm{E}}(y=0) = U^{\mathrm{E}} \\
\lim_{Y\to\infty} p^{P} &= p^{\mathrm{E}}(y=0) = P^{\mathrm{E}} \\
u_1^{\mathrm{P}}(Y=0) &= u_2^{\mathrm{P}}(Y=0) = 0
\end{aligned}$$

- Where U^{E} and P^{E} obey the Bernoulli equations

$$\partial_t U^{\mathrm{E}} + U^{\mathrm{E}} \partial_x U^{\mathrm{E}} = -\partial_x P^{\mathrm{E}}$$

Mathematical issues for the Prandtl equations

Well-posedness in suitable functional spaces:

- 2D Local existence. Monotonic in y datum.
  - Oleinik ('66): Crocco transform. Strong solutions.
  - Xin and Zhang ('04): Weak solutions for pressure of fixed sign.
  - Masmoudi-Wong ('12): energy methods + magic cancellation: the function $g = \partial_y u - u\partial_y \log(\partial_y u)$ obeys better bounds than u or $\partial_y u$.
  - In a similar spirit: Alexandre-Wang-Xu-Yang ('14).
- 2D&3D Local Existence. Analytic datum.
  - Caflisch and Sammartino ('98 - Part I): analyticity w.r.t. both x and y, exponential decay in y.
  - Cannone-Lombardo-Sammartino ('03): analyticity w.r.t. only x, exponential decay in y.
  - Kukavica-V. ('12): energy method; analyticity w.r.t. only x, any integrable decay in y.
- Local existence for non-analytic datum with critical points.
  - Gerard-Varet—Masmoudi ('13): Gevrey 7/4 initial datum with finitely many non-degenerate critical points, exponential decay in y.
  - Kukavica-Masmoudi-V.-Wong ('14): interplay between monotonicity in y an analyticity in x.
  - Xu-Zhang ('15): Sobolev initial datum which is close to a shear flow with non-degenerate critical points, algebraic decay in y.

Mathematical issues for the Prandtl equations

Ill-posedness:

- Sobolev ill-posedness: Grenier ('00), Gerard-Varet and Dormy ('09), Gerard-Varet and Nguyen ('12); Guo and Nguyen ('12)

Justify the formal derivation of the Prandtl equations in the inviscid limit, i.e. prove that

$$u^{\mathrm{NS}} = u^{\mathrm{E}}(1 - \chi_{BL}) + u^{P}\chi_{BL} + \mathcal{O}(\varepsilon)$$

- Sammartino and Caflisch ('98 - Part II): positive results in the real-analytic case
- Grenier('00); Guo-Nguyen ('12); Grenier-Guo-Nguyen ('13-'14): negative results in Sobolev spaces

Note: just because Prandtl is ill-posed (aka. strongly unstable) in some topology, it does not mean that the inviscid limit doesn't hold. The implication only goes the other way around.

Motivation

- Assume Prandtl is locally well-posed in the topology of some space X. Assume $u_0^{\text{NS}}, u_0^{\text{E}} \in X$. Does the inviscid limit hold in L^2, on an $O(1)$ time interval?
- Yes: if X is the space of real-analytic functions.
- Other settings?
- Kato-type results are conditional on the behavior of the Navier-Stokes solution: Assume assume that $u_0^{\text{E}} \in X$ and $u^{\text{NS}} \in L_t^\infty X$. Then the inviscid limit holds in L^2 for an $O(1)$ time.
- One-sided conditions à la Oleinik?
- Conditions which do not involve derivatives?

One-sided Kato criterion

Theorem (I. Constantin-Kukavica-V. ('14))

Let M_ν be a positive function which obeys

$$\int_0^T M_\nu(t)dt \to 0 \quad as \quad \nu \to 0.$$

Define the boundary layer Γ_ν by

$$\Gamma_\nu(t) = \left\{ (x_1, x_2) \in \mathbb{H} \colon 0 < x_2 \leq \frac{\nu t}{C_E} \log\left(\frac{C_E}{M_\nu(t)} \right) \right\}.$$

Assume that for all ν sufficiently small we have

$$\nu \int_0^T \left\| \left(U(x_1, t) \left(\omega^{NS}(x_1, x_2, t) + \frac{M_\nu(t)}{\nu} \right) \right)_- \right\|_{L^2(\Gamma_\nu(t))}^2 dt \leq \int_0^T M_\nu(t)dt$$

where $f_- = \min\{f, 0\}$. Then the inviscid limit holds.

- Our result states that:
  - if there is no back-flow in the Euler background, i.e. $U^{\mathrm{E}} \geq 0$, (which persists for $O(1)$ time)
  - and the part of the vorticity which is more negative than $o(1)\nu^{-1}$, i.e. $(\omega^{\mathrm{NS}} + o(1)\nu^{-1})_{-}$ in a log-Kato layer, is under control (à la Kato)
  - then the inviscid limit holds.
- Condition is strictly weaker than Kato's.
- Cannot expect the Navier-Stokes vorticity to remain of a definite sign, as the Prandtl one does.
- Result works in bounded domains with smooth boundaries.
- Condition is satisfied e.g. by viscous shear flow.

Theorem (II. Constantin-Elgindi-Ignatova-V. ('15))

Assume that there exists a constant $C_{NS} > 0$ such that

$$\sup_{\nu \in (0,\nu_0]} \int_0^T \|u^{NS}(t)\|^2_{L^\infty(\mathbb{H})} dt \leq C_{NS}\nu_0$$

and moreover that the family

$$\{u_1^{NS} u_2^{NS}\}_{\nu \in (0,\nu_0]} \quad \textit{is equicontinuous at} \quad x_2 = 0.$$

Then the inviscid limit holds in the energy norm.

- The equicontinuity condition is that there exists a function

  $$0 \leq \gamma(x_1, t) \in L^1_{t,x_1}([0,T] \times \mathbb{R})$$

  so that for any $\varepsilon > 0$, there exists $\rho = \rho(\varepsilon) > 0$ such that

  $$|u_1^{NS}(x_1, x_2, t) u_2^{NS}(x_1, x_2, t)| \leq \varepsilon \gamma(x_1, t), \quad \text{for all} \quad x_2 \in (0, \rho],$$

  and all $(t, x_1) \in [0,T] \times \mathbb{R}$, uniformly in $\nu \in (0, \nu_0]$.
- This condition implies that Lagrangian paths originating in a boundary layer do not reach in finite time beyond a fixed uniform dilate of the boundary layer. Before separation!

Theorem (III. Constantin-Elgindi-Ignatova-V. ('15))

Assume

$$\sup_{\nu\in(0,\nu_0]} \int_0^T \|u^{NS}(t)\|^2_{L^\infty(\mathbb{H})} dt \le C_{NS}\nu_0$$

and that the tangential component of the Navier-Stokes flow obeys

$$\sup_{\nu\in(0,\nu_0]} \int_0^T \|\partial_1 u_1^{NS}(t)\|^2_{L^1(\mathbb{H})} dt \le C_{NS}\nu_0$$

for some constant $C_{NS} > 0$, and that the family

$$\{\partial_1 u_1^{NS}\}_{\nu\in(0,\nu_0]} \quad \text{is uniformly integrable in} \quad L^2(0,T;L^1(\mathbb{H})),$$

Then the inviscid limit holds.

- By the last condition we mean that given an arbitrary $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that

$$\int_0^T \|\partial_1 u_1^{NS}(t)\|^2_{L^1(\Omega)} dt \le \varepsilon$$

whenever the subset $\Omega \subset \mathbb{H}$ obeys $|\Omega| \le \delta$.

- Note that $\partial_1 u_1^{NS}$ vanishes identically on $\partial\mathbb{H}$, which is not the case for the Navier-Stokes vorticity $\omega^{NS} = \partial_2 u_1^{NS} - \partial_1 u_2^{NS}$, which is expected to have a measure supported on the boundary of the domain in the inviscid limit Kelliher '(08). Thus, the vorticity is not expected to be uniformly integrable in $L_t^2 L_x^1$.
- Also, note that (uniform in ν) higher integrability of the Navier-Stokes vorticity, such as L^p for $p > 2$ cannot hold unless $U^E \equiv 0$, as is shown in Kelliher ('14).

Open problem

Removing the equicontinuity assumption on $u_1^{\mathrm{NS}} u_2^{\mathrm{NS}}$ at the boundary of the domain is an interesting question:
Q: assuming merely

$$\sup_{\nu \in (0,\nu_0]} \int_0^T \| u^{\mathrm{NS}}(t) \|_{L^\infty(\mathbb{H})}^2 dt \leq C_{\mathrm{NS}} \nu_0$$

does the inviscid limit hold?

Sketch of proof of Theorems II. and III. The Setup.

- Start like Kato: construct a boundary layer corrector such that

$$\nabla \cdot u^{K} = 0$$
$$u_1^{K}|_{\partial\mathbb{H}} = -U^{E}$$
$$u_2^{K}|_{\partial\mathbb{H}} = 0$$

- The corrector will have a characteristic length $\delta(\nu t)$, by which we mean that the following bounds hold:

$$\|u^{K}\|_{L^p(\mathbb{H})} + \|\partial_t u^{K}\|_{L^p(\mathbb{H})} + \|\partial_1 u^{K}\|_{L^p(\mathbb{H})} + \|\partial_{11} u^{K}\|_{L^p(\mathbb{H})} \leq C_{E}\delta(\nu t)^{1/p}$$
$$\|\partial_2 u_1^{K}\|_{L^p(\mathbb{H})} \leq C_{E}\delta(\nu t)^{-1+1/p}$$
$$\|\partial_1 u_2^{K}\|_{L^p(\mathbb{H})} \leq C_{E}\delta(\nu t)$$

  for all $1 \leq p \leq \infty$.
- Then the function

$$v = u^{NS} - u^{E} - u^{K}$$

  obeys $\nabla \cdot v = 0$ and $v|_{\partial\mathbb{H}} = 0$, so it is amenable to L^2 energy estimates, and

$$\lim_{\nu\to 0} \sup_{t\in[0,T]} \|v(t)\|_{L^2} = 0 \quad \Leftrightarrow \quad \lim_{\nu\to 0} \sup_{t\in[0,T]} \|u^{NS}(t) - u^{E}(t)\|_{L^2} = 0$$

Equation for v and the Prandtl equations

- The equation obeyed by v is

$$\partial_t v - \nu\Delta v + v\cdot\nabla u^{E} + u^{NS}\cdot\nabla v + \nabla q = \nu\Delta u^{E} - (\partial_t u^{K} - \nu\Delta u^{K} + u^{NS}\cdot\nabla u^{K} + u^{K}\cdot\nabla u^{E})$$

- The Prandtl equations' goal is to solve

$$\partial_t u_1^{P} - \nu\partial_{yy}u_1^{P} + (u^{P}+u^{E})\cdot\nabla u_1^{P} + u^{P}\cdot\nabla u_1^{E} = 0$$
$$u_2^{P} = -\partial_x\partial_y^{-1}u_1^{K}$$

so that in the tangential component we one is left with

$$\partial_t v - \nu\Delta v + v\cdot\nabla u^{E} + u^{NS}\cdot\nabla v + \nabla q = \nu\Delta u^{E} - v\cdot\nabla u_1^{P} - \text{small}$$

- However, the resulting term

$$\int_{\mathbb{H}} v_2\partial_2 u_1^{P} v_1 = \frac{1}{\sqrt{\nu}}\int_{\mathbb{H}} v_2\partial_Y u_1^{P} v_1 \le \frac{1}{\sqrt{\nu}}\|\partial_Y u_1^{P}\|_{L^\infty}\|v\|_{L^2}^2 \quad \text{or} \quad \le \sqrt{\nu}\|\partial_Y u_1^{P}\|_{L^\infty}\|\nabla v\|_{L^2}^2$$

is not under control: need higher order correctors

Equation for v and resulting errors

- The equation obeyed by v is

$$\partial_t v - \nu\Delta v + v\cdot\nabla u^{\mathrm{E}} + u^{\mathrm{NS}}\cdot\nabla v + \nabla q$$
$$= \nu\Delta u^{\mathrm{E}} - \left(\partial_t u^{\mathrm{K}} - \nu\Delta u^{\mathrm{K}} + u^{\mathrm{NS}}\cdot\nabla u^{\mathrm{K}} + u^{\mathrm{K}}\cdot\nabla u^{\mathrm{E}}\right)$$

- Multiply by v and integrate by parts

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 + \nu\|\nabla v\|_{L^2}^2 \leq C_{\mathrm{E}}\|v\|_{L^2}^2 + \nu C_{\mathrm{E}}\|v\|_{L^2} + T_1 + \ldots + T_6$$

where we have denoted

$$T_1 = -\int_{\mathbb{H}}(\partial_t u^{\mathrm{K}} - \nu\Delta u^{\mathrm{K}})\cdot v$$

$$T_2 + T_3 = -\int_{\mathbb{H}}(u^{\mathrm{NS}}\cdot\nabla u^{\mathrm{E}})\cdot u^{\mathrm{K}} - \int_{\mathbb{H}}(u^{\mathrm{K}}\cdot\nabla u^{\mathrm{E}})\cdot v$$

$$T_4 = -\int_{\mathbb{H}} u_1^{\mathrm{NS}} u_2^{\mathrm{NS}} \partial_1 u_2^{\mathrm{K}}$$

$$T_5 = -\int_{\mathbb{H}} \left((u_1^{\mathrm{NS}})^2 - (u_2^{\mathrm{NS}})^2\right)\partial_1 u_1^{\mathrm{K}}$$

$$T_6 = -\int_{\mathbb{H}} u_1^{\mathrm{NS}} u_2^{\mathrm{NS}} \partial_2 u_1^{\mathrm{K}}$$

Construction of the corrector u^{κ}

- Eliminate the contribution from T_1 to leading order in ν:

$$u_1^{\kappa}(x_1, x_2, t) = -U^{E}(x_1, t)\left(\operatorname{erfc}\left(\frac{x_2}{\sqrt{4\nu t}}\right) - \sqrt{4\nu t}\,\eta(x_2)\right)$$

$$u_2^{\kappa}(x_1, x_2, t) = -\int_0^{x_2} \partial_1 u_1^{\kappa}(x_1, y, t)dy$$

  where η is a positive bump, of mass $1/\sqrt{\pi}$, approximating $\chi_{[1,2]}$, and $\operatorname{erfc}(z) = 1 - \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} \exp(-y^2)dy$.
- Note: essential that u_1^{κ} has zero mean in x_2.
- This has the characteristic length of the Prandtl layer

$$\delta(\nu t) = \sqrt{\nu t}$$

Bounding T_5

- Assuming an $L_t^2 L_x^\infty$ bound on u^{NS}, we may estimate

$$\int_0^T |T_5(t)|dt \le \int_0^T \int_{\mathbb{H}} \left|\left((u_1^{\text{NS}})^2 - (u_2^{\text{NS}})^2\right) \partial_1 u_1^{\text{K}}\right|$$
$$\le \|u^{\text{NS}}\|_{L^2(0,T;L^\infty)}^2 \|\partial_1 u_1^{\text{K}}\|_{L^\infty(0,T;L^1)}$$
$$\le (C_{\text{NS}}\nu_0)^2 C_{\text{E}} (\nu T)^{1/2}$$

- For this term a weaker assumption would have been OK:

$$u^{\text{NS}} \quad \text{uniformly bounded in} \quad L^1(0,T; L_{x_1}^2 L_{x_2}^p(\mathbb{H}))$$

for any $p > 2$.

Bounding T_6

- We estimate

$$|T_6(t)| \leq C_{\mathrm{E}}(\nu t)^{1/2} + C|T_{6,\nu}(t)|$$

where

$$\int_0^T |T_{6,\nu}(t)| dt$$
$$= \int_0^T \int_{\mathbb{H}} \left| u_1^{\mathrm{NS}}(x_1, \sqrt{4\nu t} y, t) u_2^{\mathrm{NS}}(x_1, \sqrt{4\nu t} y, t) \right| |U^{\mathrm{E}}(x_1, t)| \exp(-y^2) dx_1 dy dt$$

- The measure

$$\mu_{x_1,y,t} = \|U^{\mathrm{E}}(x_1, \cdot)\|_{L^\infty([0,T])} \exp(-y^2) dx_1 dy dt$$

gives bounded mass to $[0, T] \times \mathbb{H}$.

- If $\int_0^T \sup_\nu \|u^{\mathrm{NS}}(t)\|_{L^\infty}^2 dt < \infty$, may conclude by DCT if we knew

$$u_1^{\mathrm{NS}}(x_1, \sqrt{4\nu t} y, t) u_2^{\mathrm{NS}}(x_1, \sqrt{4\nu t} y, t) \to 0 \quad \text{as} \quad \nu \to 0$$

pointwise(!) in (x_1, y, t).

T_6 bound in Theorem II.

- Assume equicontinuity at $x_2 = 0$ of the family $u_1^{\text{NS}} u_2^{\text{NS}}$.
- Given $\varepsilon > 0$, let $\rho(\varepsilon) > 0$ be such that: [def of equicontinuity].

$$\begin{aligned}
&\int_0^T |T_{6,\nu}(t)|dt \quad \text{(split into } y \geq \frac{\rho}{\sqrt{4\nu t}} \text{ and } y \leq \frac{\rho}{\sqrt{4\nu t}}\text{)} \\
&\leq \|U^{\text{E}}\|_{L^\infty(0,T;L^1_{x_1}(\mathbb{R}))} \int_0^T \|u^{\text{NS}}(t)\|^2_{L^\infty_{x_1,x_2}(\mathbb{H})} \left(\int_{y \geq \frac{\rho}{\sqrt{4\nu t}}} \exp(-y^2) dy \right) dt \\
&\quad + \|U^{\text{E}}\|_{L^\infty(0,T;L^\infty_{x_1}(\mathbb{R}))} \int_0^T \int_{y \leq \frac{\rho}{\sqrt{4\nu t}}} \varepsilon\, \gamma(x_1, t) \exp(-y^2) dx_1\, dy dt \\
&\leq C_{\text{E}} C_{\text{NS}} \nu_0 \operatorname{erfc}\left(\frac{\rho}{\sqrt{4\nu T}} \right) + \varepsilon C_{\text{E}} \|\gamma\|_{L^1(0,T;L^1(\mathbb{R}_+))}
\end{aligned}$$

- Passing $\nu \to 0$ with $\rho(\varepsilon)$ and T are fixed, and $\operatorname{erfc}(z) \to 0$ as $z \to \infty$, we arrive at

$$\lim_{\nu \to 0} \int_0^T |T_{6,\nu}(t)|dt \leq \varepsilon \|U^{\text{E}}\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \|\gamma\|_{L^1(0,T;L^1(\mathbb{R}_+))}.$$

- Recall γ is independent of ε, and $\varepsilon > 0$ is arbitrary.

T_6 bound in Theorem III.

- Assuming the uniform boundedness of and uniform integrability of $\partial_1 u_1^{\text{NS}}$ in $L^2(0,T;L^1(\mathbb{H}))$ we have:

$$\int_0^T |T_{6,\nu}(t)|dt$$
$$\leq \int_{\mathbb{R}_+} \exp(-y^2)$$
$$\times \int_0^T \int_{\mathbb{R}} \|u_1^{\text{NS}}(t)\|_{L^\infty_{x_1,x_2}(\mathbb{H})} \int_0^{\sqrt{4\nu t}y} |\partial_2 u_2^{\text{NS}}(x_1,z,t)|dz \|U^{\text{E}}(x_1)\|_{L^\infty_t[0,T]} dx_1 dt dy$$
$$\leq \|u_1^{\text{NS}}(t)\|_{L^2(0,T;L^\infty(\mathbb{H}))} \int_{\mathbb{R}_+} B_\nu(y)\exp(-y^2)dy$$

where

$$(B_\nu(y))^2 = \int_0^T \left(\int_{\mathbb{H}} |\partial_1 u_1^{\text{NS}}(x_1,z,t)| \|U^{\text{E}}(x_1)\|_{L^\infty([0,T])} \mathbf{1}_{z\leq\sqrt{4\nu T}y} dz dx_1\right)^2 dt$$

- Pointwise, we have

$$B_\nu(y) \leq \|U^{\text{E}}\|_{L^\infty(0,T;L^\infty(\mathbb{R}))} \|\partial_1 u_1^{\text{NS}}\|_{L^2(0,T;L^1_{x_1,z}(\mathbb{H}))}$$
$$\leq C_{\text{E}} C_{\text{NS}} \nu_0 \in L^1(\exp(-y^2)dy).$$

T_6 bound in Theorem III.

- In order to apply DCT and conclude that

$$\lim_{\nu\to 0} \int_{\mathbb{R}_+} B_\nu(y) \exp(-y^2) dy = 0$$

we need to show that for each fixed $y > 0$ we have

$$\lim_{\nu\to 0} B_\nu(y) = 0.$$

- Fix $\varepsilon > 0$, and pick the $\eta = \eta(\varepsilon/2C_{\rm E})$ given by uniform integrability.
- For $R > 0$, define the level set

$$A_R = \{x_1 \in \mathbb{R} \colon \|U^{\rm E}(x_1)\|_{L^\infty[0,T]} \le R\}.$$

- If R is sufficiently small (depending on ε) we use

$$\|\partial_1 u_1^{\rm NS}(x_1, z, t) \mathbf{1}_{z\le\sqrt{4\nu T}y} \|U^{\rm E}(x_1)\|_{L^\infty[0,T]}\|_{L^2(0,T;L^1_{x_1,z}(A_R\times\mathbb{R}_+))} \le R C_{\rm NS} \nu_0 C_{\rm E}$$

- On the other hand, $|A_R^c| \le \frac{C_{\rm E}}{R}$ so that

$$|\Omega| = |A_R^c \times \{z \colon 0 < z \le \sqrt{4\nu T}y\}| \le \frac{C_{\rm E}}{R}\sqrt{4\nu T}y \le \eta(\varepsilon/2C_{\rm E})$$

if we choose ν sufficiently small (depending on y and ε).

Van Dommelen and Shen ('80) - Prandtl separation

- Consider a non-trivial stationary Euler flow at infinity (i.e. non-constant, or at least constant not 0)

$$U^{E}(x) = \kappa \sin(x)$$
$$-\partial_x P^{E}(x) = \frac{\kappa^2}{2}\sin(2x)$$

where $\kappa \in \mathbb{R}$ is a parameter, and $x \in [-\pi, \pi]$.

- These are stationary solutions of the Bernoulli equation

$$\partial_t U^{E} + U^{E}\partial_x U^{E} = -\partial_x P^{E}.$$

- Consider the Prandtl equations with these boundary conditions.
- Conjecture: Based on numerical experiments, the Prandtl solution cannot remain smooth for all time, i.e. they blow up in finite time.

Van Dommelen and Shen ('80)

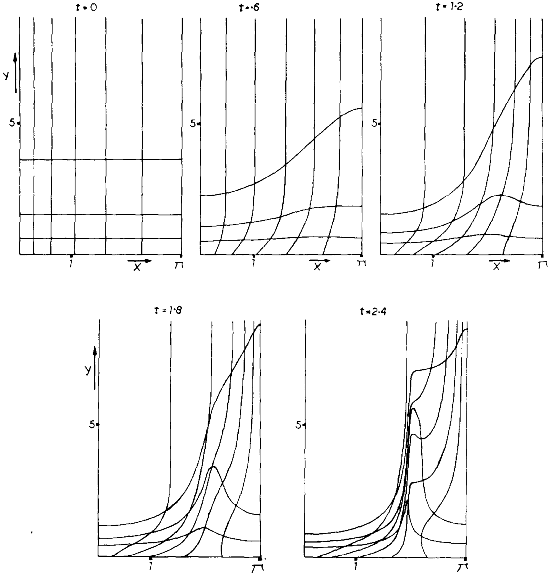

Figure: The distortion of a typical Lagrangian grid with time.

Van Dommelen and Shen ('80)

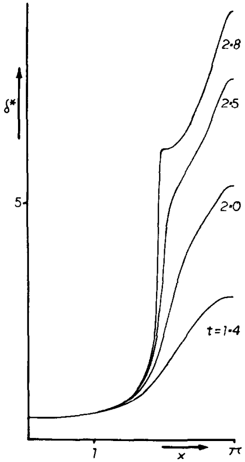

Figure: The variation of the displacement thickness $\delta^*(t, x)$ for various times.

$$\delta^*(t,x) = \int_0^\infty \left(1 - \frac{u_1^{\mathrm{P}}(x,y,t)}{U^{\mathrm{E}}(x,t)}\right)$$

Blowup in Prandtl?

- The "numerical blowup" seen by van Dommelen and Shen was reconfirmed by several groups, on finer computers with more sophisticated methods:
  - Cassel-Smith-Walker ('96)
  - Hong-Hunter ('03)
  - Gargano-Sammartino-Sciacca ('09)
  - Caflisch-Gargano-Sammartino-Sciacca ('15)
- Goal: a mathematically rigorous proof?
- E-Engquist ('01): consider $\kappa = 0$, and datum that is compactly supported in y, which is large (just about to blow up), and show that it does indeed blow up in finite time.
- Proof by contradiction: either smoothness or decay towards 0 as $y \to \infty$ fails. Local existence in this class missing at the time.
- Caflisch-Gargano-Sammartino-Sciacca ('15): at the level of numerical simulations, the complex structure of the $\kappa = 0$ blowup is of different type from the $\kappa \neq 0$ singularity.
- Dalibard-Masmoudi ('14): proof of separation in a steady flow.

Theorem (IV. Kukavica-V.-Wang ('15))

Consider the Cauchy problem for the Prandtl equations with boundary conditions at $y = \infty$ matching the van Dommelen-Shen scenario. There exists an open set of initial conditions u_0^P which are real-analytic in x and y, such that the unique real-analytic solution u^P to the Prandtl equations, blows up in finite time.

- Who blows up?

$$\mathcal{G}(t) = \int_0^\infty (\kappa\varphi(t,y) - \partial_x u_1^P(t,0,y))w(y)dy$$

where $\varphi(t,y)$ is a suitable caloric lift of the boundary conditions, and $w(y)$ is a suitable integrable weight.

- This is approximatively: the displacement thickness

$$\kappa\delta^*(t,0) = \lim_{x\to 0}\int_0^\infty \left(\kappa - \frac{u^P(t,x,y)}{\sin(x)}\right) dy = \int_0^\infty (\kappa - \partial_x u^P(t,0,y))\, dy$$

Remarks

Inviscid limit

- Local in time inviscid limit holds for these initial conditions Sammartino-Caflisch ('98)
- We prove that the Prandtl expansion approach to the inviscid limit should only be expected to hold on finite time intervals.

More general Euler flows

- The proof holds if $U^E(x) = \kappa \sin(x)$ is replaced by any odd function of x, upon letting $P^E(x) = -(U^E(x))^2/2$.

More general initial conditions

- The analyticity of the initial datum is only used to ensure the local existence and uniqueness of (sufficiently) strong solutions.
- Instead, we may pick any initial datum which is matches the $\kappa \sin(x)$ at $y = \infty$, and for which the Prandtl system is locally well-posed.

Size of the datum: needs to be sufficiently large, depending on κ.

- Ignatova-V. ('15): ε-small analytic perturbations of the error function solve Prandtl for $[0, T_\varepsilon]$, where $T_\varepsilon \geq \exp(\varepsilon^{-1}/\log(\varepsilon^{-1}))$.

Sketch of proof of Theorem IV

- Consider datum which is smooth and odd with respect to x.
- The unique smooth solution obeys the same symmetry, and thus

$$u(t,0,y) = \partial_y u(t,0,y) = \partial_{xx} u(t,0,y) = 0.$$

- Restrict dynamics to the x-axis, where the Lagrangian trajectories are frozen, and the vorticity is vanishing identically.
- The function

$$b(t,y) = (-\partial_x u)(t,0,y)$$

obeys

$$\partial_t b - \partial_{yy} b = b^2 - \partial_y^{-1} b \, \partial_y b - \kappa^2$$
$$b|_{y=0} = 0, \qquad b|_{y=\infty} = -\kappa.$$

- Do not like: b doesn't have a definite sign; there is a competition between b^2 and $-\kappa^2$ on the RHS.

Sketch of proof of Theorem IV (cont'd)

- Lift the function b "up" by an artificial corrector

$$a(t,y) = b(t,y) + \varphi(t,y)$$

where

$$\partial_t \varphi - \partial_{yy}\varphi = \kappa^2$$
$$\varphi|_{y=0} = 0, \qquad \varphi|_{y=\infty} = \kappa + \kappa^2 t$$
$$\varphi|_{t=0} = \kappa \mathrm{Erf}(y/2).$$

so that $a(t,y)$ obeys

$$\partial_t a - \partial_{yy} a = a^2 - \partial_y^{-1} a\, \partial_y a + L_\varphi[a] + F_\varphi$$
$$a|_{y=0} = 0, \qquad a|_{y=\infty} = \kappa^2 t \geq 0$$
$$L_\varphi[a] = -2a\varphi + \partial_y^{-1} a\, \partial_y \varphi + \partial_y^{-1}\varphi\, \partial_y a$$
$$F_\varphi = \varphi^2 - \partial_y^{-1}\varphi\, \partial_y \varphi \geq 0.$$

- The upshot: minimum principle for a

$$a_0(y) \geq 0 \Rightarrow a(t,y) \geq 0.$$

Sketch of proof of Theorem IV (cont'd)

- Define a Lyapunov functional

$$\mathcal{G}(t) = \int_0^\infty a(t,y)w(y)dy$$

where w is a non-negative weight, with $w \in L^1 \cap W^{2,\infty}$, and $w(y=0) = w(y=\infty) = 0$.

- Then, by choosing w very carefully, we may prove

$$\frac{d\mathcal{G}}{dt} = \int a\partial_{yy}w + 2\int a^2 w - \frac{1}{2}\int (\partial_y^{-1}a)^2\partial_{yy}w + \int L_\varphi[a]w + \int Fw$$
$$\geq \frac{1}{c_*}\mathcal{G}^2 - c_*(1+t)\mathcal{G}$$

for some $c_* = c_*(\kappa, w) > 0$.

- To conclude, choose

$$\mathcal{G}_0 \geq 4c_*^2$$

and obtain the finite time blowup of $\mathcal{G}(t)$.

- For example, let $A \gg 1$, and define

$$u_0(x,y) = (\kappa \mathrm{Erf}(y/2) - a_0(y))\sin(x), \qquad a_0(y) = Ay^2\exp(-y^2).$$