Vortex layers of small thickness

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Vorticity

Consider the incompressible Euler equations in 2D:

$$\begin{aligned} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \nabla p &= 0 \\ \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 \, . \end{aligned}$$

If one defines the vorticity:

$$\omega = \boldsymbol{\nabla} \times u$$

then Euler equations, in the vorticity formulation, write:

$$\begin{aligned} \partial_t \omega + \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega &= 0 \\ \omega(x, y, t = 0) &= \omega_0 \end{aligned}$$

To recover $\boldsymbol{u}$, introduce the stream function ψ such that $\boldsymbol{u} = \boldsymbol{\nabla}^{\perp} \psi$. Then:

$$\Delta \psi = -\omega$$

and therefore:

$$\boldsymbol{u} = -\boldsymbol{\nabla}^{\perp} \Delta^{-1} \omega$$

Smooth and less regular solutions

Existence and uniqueness of solutions for given initial data:

When the initial datum ω_0 is *regular* one can achieve many well posedness results of the above problem :

Global in time $\exists$ of $2D$ classical solutions $\Rightarrow$ $\boxed{\text{well-posedness à la Hadamard}}$

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However there are cases in which one is interested in initial data where vorticity has less regular configuration.

Non-smooth vorticity

Weak solutions appropriate for modeling an isolated region of intense vorticity, e.g. vorticity discontinuous but bounded:

- Yudovich (1963): solutions for $\omega_0 \in L^1 \bigcap L^\infty$;
- From the above result, it follows an existence and uniqueness result globally in time for vortex-patches initial data. M& B book
- Zero Viscosity limit Constantin & Wu 1995, Sueur 2015

Vortex sheets

These results DO NOT include less regular initial data as measure-valued initial vorticity.

The interest in these kind of data, beside the intrinsic interest, is also motivated by the so called vortex–sheet datum.

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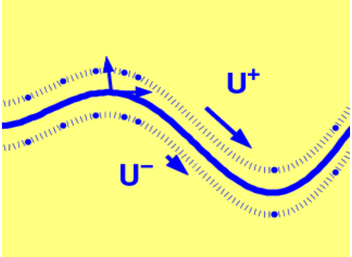

Vortex sheet

- a vortex sheet is curve on which vorticity is highly concentrated as a δ function
- outside the curve, the flow is irrotational

It is an interface across which the tangential (to the curve) velocity experiences a discontinuity.

Vortex sheets

The importance of vortex sheets flows is due to:

- a model for the wake left behind from a body immersed in a flow at large Reynolds number;
- prototype of interface dynamics
- mixing layers of fluids.

Moreover (with a little of hindsight) the complicated evolution of vortex sheets is a natural source for the spontaneous appearance of small scale motion in incompressible fluids (onset of turbulence).

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In these cases the interest is:

1. determination of the motion of the curve,
2. occurrence of singularity;
3. characterization of singularity,
4. ...

Initial value problem for Euler eqs with VS initial data

The basic result is of the following form:

Existence of solutions

If ω_0 is a bounded measure with positive singular part, then Euler equations (in the weak form) admit as solution a bounded measure ω_t, and $\boldsymbol{u} \in L^\infty_{loc}(\mathbb{R}, L^2_{loc})$.

Delort '91, Majda '93, Di Perna and Majda '87; Chemin '95; Evans and Muller 94'; Schochet '95; Lopes, Nussenzveig and Xin '01, '06; Niu, Jiu and Xin '07; Brenier, De Lellis, and Szekelyhidi '10

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But:

- it gives no information on the structure of the solution.

The explicit approach

One can tackle the mathematical description of VS from a different point a view: one tries to explicitly determine the interface using a time-dependent parametrization.

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Let $y = \varphi(x)$ a curve where the vorticity is initially concentrated:

$$\boxed{\omega_0(x, y) = \gamma_0(x)\delta(y - \varphi(x))}$$

$\gamma_0(x)$ is the vorticity density, i.e. the jump strength.

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Assuming that a vortex sheet remains a vortex sheet one writes:

$$\boxed{\omega(x, y, t) = \gamma(x, t)\delta(y - \varphi(x, t))}$$

and is interested in finding the $y = \varphi(x, t)$.

The Birkhoff–Rott equation

- Characterize a point in the plane (x, y) by the complex variable $z = x + iy$.
- Parametrize the curve using the arc length s:

$$x + i\varphi(x, t) = Z(s, t)$$

- Use the potential theory to write the velocity field induced by the vorticity concentrated on the sheet:

$$u - iv = -\frac{i}{2\pi} \int \frac{\gamma(s, t) ds'}{z - Z(s', t)}$$

- The velocity of the sheet is the average of the velocities above and below the sheet:

$$U - iV = -\frac{i}{2\pi} PV \int \frac{\gamma(s, t) ds'}{Z(s, t) - Z(s', t)}$$

The Birkhoff–Rott equation

Finally characterize a point P the curve using (instead of s) the total circulation Γ contained between a reference point $\bar{P}$ and P:

BR equation

$$\frac{\partial Z^*}{\partial t} = -\frac{i}{2\pi} PV \int \frac{d\Gamma'}{Z(\Gamma, t) - Z(\Gamma', t)}$$

Nonlinear, singular integro-differential equation.

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A posteriori one justifies this equation showing that the velocity field induced by the vorticity concentrated on this curve, whose strength is determined using the conservation of vorticity, satisfies the Euler equations in the weak form.

B–R equation: an alternative formulation

Use cartesian coordinates (x, y). Let $\varphi(x, t)$ the graph of the VS, and let $\gamma(x, t)$ the strength of the VS (the jump in the tangential velocity); i.e.:

$$\omega(x, y, t) = \gamma(x, t)\delta(y - \varphi(x, t))\,.$$

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Using elementary considerations (the fact that the curve $\varphi(x,t)$ is *material* and conservation of vorticity) one gets:

BR equation: Alternative form

$$\begin{aligned} \partial_t\varphi + U\partial_x\varphi &= V \\ \partial_t\gamma + \partial_x(\gamma U) &= 0\,. \end{aligned}$$

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where (U, V) is the velocity field on the curve:

$$U = -\frac{1}{2\pi} PV \int \frac{\varphi(x,t)-\varphi(x',t)}{(x-x')^2 + (\varphi(x,t)-\varphi(x',t))^2}\gamma(x')\,dx' \qquad (1)$$

$$V = \frac{1}{2\pi} PV \int \frac{x-x'}{(x-x')^2 + (\varphi(x,t)-\varphi(x',t))^2}\gamma(x')\,dx' \qquad (2)$$

Some of the known fact about BR equation

The BR equations develop singularity (infinite curvature)

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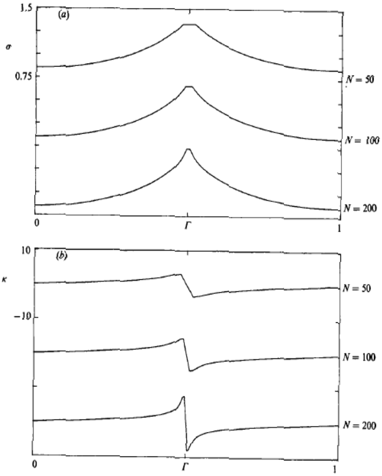

Although smooth initial data in 2D are known to lead to smooth solutions for all time, this result shows that singular initial data (a vortex sheet in 2D) can become more singular (infinite curvature of the sheet) in finite time.

Some of the known fact about BR equation

- Kelvin-Helmholtz instability:

$$Z(\Gamma, t) = \Gamma$$ **VS of constant strenght**

is an equilibrium solution of the BR equation.
Linear stability analysis of this equilibrium yields a growing eigenfunction:

$$(\, Z(\Gamma, t) - \Gamma \,) \quad \propto \quad \exp\left(\pi k t\right) \sin\left(2\pi k \Gamma\right)$$

Short wavelength solutions with arbitrary large growth rates.

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Some of the known fact about BR equation

- Use of analytic functions provides the stabilization necessary to rigorously construct solutions in the presence of physical instabilities.
  This was first conjectured by Birkhoff(1962) and then rigorously proved:

Theorem (Sulem, Sulem, Bardos, Frisch 81')

If the initial data are such that the Fourier modes are exponentially decaying (analytic data), then the B–R are well posed (short time existence)

The proof holds for both the $2D$ and the $3D$ case.

Some of the known fact about B–R equation

Long Time Existence of Solutions:

Theorem (Duchon and Robert '88, Caflisch and Orellana '86)

Suppose that initially the vortex sheet has a small sinusoidal perturbation, so that

$$Z(y,t) = y + i\varepsilon \sin y \qquad \varepsilon \ \textit{small}$$

Then the vortex sheet equation has a smooth solution for a time interval $0 < t < 2K|\log \varepsilon|$ *in which* $K < 1$ *and* $K \to 1$ *as* $\varepsilon \to 0$*.*

The time of existence is nearly optimal, since asymptotic analysis (Moore 1984) indicates that a singularity will form at the critical time $t = 2|log\varepsilon| + O(log|log\varepsilon|)\cdot$

Some of the known fact about B–R equation

BR-α equations:

- $2D$-Euler-α equations: existence and uniqueness of a global weak solution with initial vorticity Radon measure on $\mathbb{R}^2$, with a unique lagrangian flow map of the particles.
- Bardos, Linshiz, and Titi 2010 proved a long time existence and uniqueness theorem in Hölder spaces for the BR-α equations

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  The α-regularization mollifies the Kelvin-Helmholtz instability: Linearization about the flat sheet of uniform intensity γ_0 gives the following growth rate for the Fourier modes:

$$\lambda(k) \sim |k|\left(1 - (1 + \frac{1}{\alpha^2 k^2})^{-\frac{1}{2}}\right)$$

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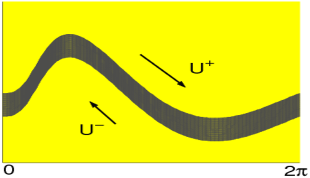
Wu, 2005 using also a result of David, 1984, gave a weaker notion of the solution of the BR equation, chord-arc curves, that is able to follow the roll-up of the sheet after the singularity.

Vortex Layers

VISCOSITY DIFFUSES VORTICITY

To model this phenomenon one can use a layer of uniform vorticity and small thickness:

Vortex Layers:



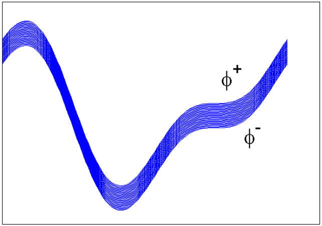
Vortex Layers of uniform vorticity

1. Moore '78: Using asymptotic analysis he derived the following correction to the Birkhoff–Rott equation:

$$\frac{\partial Z^*}{\partial t} = -\frac{i}{2\pi} PV \int \frac{d\Gamma'}{Z(\Gamma, t) - Z(\Gamma', t)} - \epsilon \frac{i}{6\bar{\omega}} \frac{\partial}{\partial \Gamma} \left(U^4 \frac{\partial Z^*}{\partial \Gamma} \right) + O(\varepsilon^2)$$

   where $U(\Gamma, t) = \gamma(s, t)$ and $\bar{\omega} = \gamma H$ is the total vorticity contained in each cross section;
2. Baker and Shelley, '90: Numerical results;
3. Benedetto and Pulvirenti, '92 proved rigorously that the dynamics of a vortex layer (of uniform vorticity) converges to the dynamics of a vortex sheet when the thickness goes to zero.

Denoting with φ^+ and φ^- the boundaries of the domain where the vorticity is concentrated they derived the following equations:



$$
\begin{aligned}
\partial_t \varphi^+ + U^+ \partial_x \varphi^+ &= V^+ \\
\partial_t \varphi^- + U^- \partial_x \varphi^- &= V^- \\
\partial_t \gamma + \partial_x \left(\frac{1}{\varepsilon} \int_{\varphi^-}^{\varphi^+} u(x,y)dy \right) &= 0\,.
\end{aligned}
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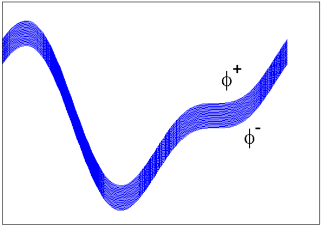
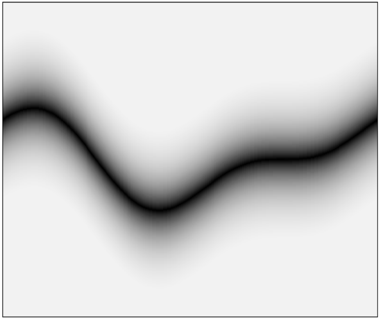


$$\begin{aligned} \partial_t \varphi^+ + U^+ \partial_x \varphi^+ &= V^+ \\ \partial_t \varphi^- + U^- \partial_x \varphi^- &= V^- \\ \partial_t \gamma + \partial_x \left(\frac{1}{\varepsilon} \int_{\varphi^-}^{\varphi^+} u(x,y) dy \right) &= 0\,. \end{aligned}$$

$$\boldsymbol{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2\pi\varepsilon} \int_{\infty}^{\infty} dx' \int_{\varphi^-}^{\varphi^+} \frac{\begin{pmatrix} y'-y \\ x-x' \end{pmatrix}}{(x-x')^2 + (y-y')^2} \, dy'$$

Our goal

We want to deal with the situation when vorticity is not compactly supported.



It is exponentially decaying outside a small layer: a different approach is needed. One has to deal with the "full" problem

Statement of the problem

In the periodic strip $(x, y) = [-\pi, \pi[\times\mathbb{R}$ we consider Euler equations

$$\begin{aligned} \partial_t \omega^\varepsilon + \boldsymbol{u}^\varepsilon \cdot \boldsymbol{\nabla} \omega^\varepsilon &= 0 \\ \boldsymbol{u}^\varepsilon &= \boldsymbol{\nabla}^\perp \Delta^{-1} \omega^\varepsilon \\ \omega^\varepsilon(x, y, t = 0) &= \omega_0^\varepsilon(x, y) \end{aligned}$$

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There exist a curve $\varphi_0(x)$ and two constant c, μ_0 s.t.:

$$\sup_y \|\omega_0^\varepsilon\|_x < c\varepsilon^{-1} e^{-\mu_0 \varepsilon^{-1} |y - \varphi_0(x)|}$$

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$$\sup_y \|\omega_0^\varepsilon\|_x < c\varepsilon^{-1} e^{-\mu_0 \varepsilon^{-1} |y - \varphi_0(x)|}$$

and the total amount of vorticity of the layer does not depend on ε, i.e.:

$$\int \omega_0^\varepsilon(x, y)\, dx\, dy = O(1)$$

Statement of the problem: Functional setting

Denote by D_ρ the strip of the complex plane of width ρ:

$$D_\rho \equiv \{(x,\eta) : \ x \in \mathbb{R}/\pi\mathbb{Z}, |\eta| < \rho\} .$$

and with $\Sigma(\sigma)$, where $0 < \sigma < \pi/4$, the cone in the complex plane:

$$\Sigma(\sigma) \quad \equiv \quad \{(Y,\lambda) : \ Y \in \mathbb{R}, \ |\lambda| < |Y| \tan\sigma\} \tag{3}$$

Moreover let α be a real number such that $0 < \alpha < 1$.
For a function $f : D_\rho \to \mathbb{C}$ we introduce the notation:

$$|f|_\rho \quad \equiv \quad \sup_{(x,\eta)\in D_\rho} |f(x+i\eta)| \ ,$$

$$|f|_\rho^{(\alpha)} \quad \equiv \quad \sup_{(x,\eta),\,(\bar{x},\eta)\in D_\rho} \frac{|f(x+i\eta) - f(\bar{x}+i\eta)|}{|x-\bar{x}|^\alpha}$$

For a function $g : D_\rho \times \Sigma(\sigma) \to \mathbb{C}$ we introduce the notation:

$$|g|_{\rho,\sigma} \quad \equiv \quad \sup_{(x,\eta)\in D_\rho,\ (y,\lambda)\in\Sigma(\sigma)} |g(x+i\eta, y+i\lambda)| \ ,$$

$$|g|_{\rho,\sigma}^{(\alpha)} \quad \equiv \quad \sup_{\substack{(x,\eta),\ (\bar{x},\eta)\in D_\rho \\ (y,\lambda),\ (\bar{y},\lambda)\in\Sigma(\sigma)}} \frac{|g(x+i\eta, y+i\lambda) - g(\bar{x}+i\eta, \bar{y}+i\lambda)|}{\left[(x-\bar{x})^2 + (y-\bar{y})^2\right]^{\alpha/2}}$$

Statement of the problem: Functional setting

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Let $f : D_\rho \to \mathbb{C}$. Then se say $f \in B_\rho$ when:

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Definition

Let $f : D_\rho \to \mathbb{C}$. then we say $f \in B_{m,\rho}$ when:

$$\|f\|_{m,\rho} \equiv \sum_{j \leq m} |\partial_x^j f|_\rho + |\partial_x^m f|_{l,\rho}^{(\alpha)} < \infty$$

Statement of the problem: Functional setting

Definition

Let $g : D_\rho \times D_\sigma \to \mathbb{C}$. Then we say $g \in B_{m,\rho,\sigma,\mu}$ when:

$$\|g\|_{m,\rho,\sigma,\mu} = \sum_{i+j \le m} |e^{\mu y} \partial_x^i \partial_y^j g|_{\rho,\sigma} + \sum_{i+j=m} |e^{\mu y} \partial_x^i \partial_y^j g|_{\rho,\sigma}^{(\alpha)} < \infty$$

Definition

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$$\begin{aligned} \|g\|_{m,\rho,\sigma,\mu,\beta,T} \quad = \quad & \sum_{i+j \le m} \sup_{0 \le t \le T} |e^{(\mu - \beta t) y} \partial_x^i \partial_y^j g(\cdot, \cdot, t)|_{\rho - \beta t, \sigma - \beta t} + \\ & \sum_{i+j=m} \sup_{0 \le t \le T} |e^{(\mu - \beta t) y} \partial_x^i \partial_y^j g|_{\rho - \beta t, \sigma - \beta t}^{(\alpha)} < \infty \end{aligned}$$

A preliminary estimate

The first step is to prove that the norm of velocity field is bounded by the norm of the vorticity.

One has that:

$$\boxed{\boldsymbol{u} = \boldsymbol{K} * \omega}$$

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$$\begin{aligned} K_u(x,y) &= \frac{1}{8\pi^2}\frac{\sinh(2y)}{\sin^2(x)+\sinh^2(y)} \\ K_v(x,y) &= -\frac{1}{8\pi^2}\frac{\sin(2x)}{\sin^2(x)+\sinh^2(y)} \end{aligned}$$

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$$

Just the analogous in the periodic strip $[-\pi,\pi[\times\mathbb{R}$ of what one would have in the whole plane $\mathbf{R}^2$:

$$
\begin{aligned}
K_u &= \frac{1}{2\pi}\frac{y}{x^2+y^2} \\
K_v &= -\frac{1}{2\pi}\frac{x}{x^2+y^2}
\end{aligned}
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A preliminary estimate

Therefore we can write the velocity field in the following way:

$$u(x+i\eta_1, y+i\eta_2) = \int_{x-\frac{\pi}{2}}^{x+\frac{\pi}{2}} \int_{-\infty}^{\infty} K_u(x-x', y-y')\omega\left(x'+i\eta_1, y'+i\eta_2\right) dx' dy'$$

$$v(x+i\eta_1, y+i\eta_2) = \int_{x-\pi/2}^{x+\pi/2} \int_{-\infty}^{\infty} K_v(x-x', y-y')\omega\left(x'+i\eta_1, y'+i\eta_2\right) dx' dy'$$

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Therefore we can write the velocity field in the following way:

$$u(x+i\eta_1, y+i\eta_2) = \int_{x-\frac{\pi}{2}}^{x+\frac{\pi}{2}} \int_{-\infty}^{\infty} K_u(x-x', y-y')\omega\left(x'+i\eta_1, y'+i\eta_2\right)\, dx'dy'$$

$$v(x+i\eta_1, y+i\eta_2) = \int_{x-\pi/2}^{x+\pi/2} \int_{-\infty}^{\infty} K_v(x-x', y-y')\omega\left(x'+i\eta_1, y'+i\eta_2\right)\, dx'dy'$$

Proposition: Potential estimate for highly concentrated vorticity

Let u and v be expressed by the above formulas, and let $\omega \in B_{1,\rho,\sigma,\mu}$. Then $u \in B_{1,\rho,\sigma}$, $v \in B_{1,\rho,\sigma}$, and the following estimates hold:

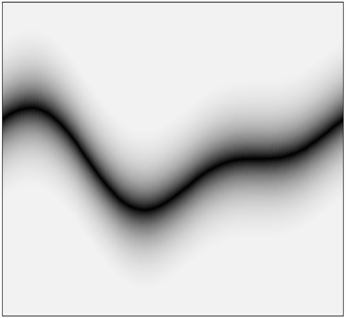
$$\begin{aligned} \|u\|_{1,\rho,\sigma}^{(\alpha)} &\leq c\varepsilon\|\omega\|_{1,\rho,\sigma,\mu}^{(\alpha)} \\ \|v\|_{1,\rho,\sigma}^{(\alpha)} &\leq c\varepsilon\|\omega\|_{1,\rho,\sigma,\mu}^{(\alpha)} \end{aligned}$$

A preliminary difficulty

If one looks at the equation for the vorticity:

$$\boxed{\partial_t \omega + u \partial_x \omega + v \partial_y \omega = 0}$$

One sees that all terms are $O(\varepsilon^{-1})$.



The cartesian reference frame $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$ is not the appropriate one to handle the vortex layer. One needs a reference frame which can separate rapid $O(\varepsilon^{-1})$ variations from slow $O(1)$ variations.

Comoving frame

We want to write the Euler equations in the comoving reference frame. To be more precise we make the change of coordinates $(x, y, t) \to (\xi, Y, \tau)$ defined as:

$$x = \xi + X(\xi, \tau)\,, y = \varepsilon Y + \varphi(x(\xi, Y = 0, \tau), \tau)\,, \tau = t\ .$$

where the *shift factor* X is defined as:

$$X(\xi, \tau) = \int_0^\tau u(x(\xi, Y = 0, \tau'), Y = 0, \tau')d\tau'$$

Notice how the new coordinate system is not orthogonal. Defining the rescaled vorticity $\tilde{\omega} = \varepsilon\omega$

Euler equations:

$$\partial_\tau \tilde{\omega} + \frac{(u - u^\varphi)}{[1 + X_\xi(\xi, \tau)]}\partial_\xi \tilde{\omega} + \frac{1}{\varepsilon}\left[-\partial_\xi \varphi \frac{(u - u^\varphi)}{[1 + X_\xi(\xi, \tau)]} + (v - v^\varphi)\right]\partial_Y \tilde{\omega} = 0\ .$$

Euler equation: final form

We use the incompressibility condition

$$\frac{\partial_\xi u}{1+X_\xi} + \frac{1}{\varepsilon}\left[-\frac{\partial_\xi \varphi}{1+X_\xi}\partial_Y u + \partial_Y v\right] = 0 \ .$$

to re-write the Euler equations in the form:

$$\partial_\tau \tilde{\omega} + \frac{1}{[1+X_\xi(\xi,\tau)]}\left[(u-u^\varphi)\,\partial_\xi \tilde{\omega} - \int_0^Y \partial_\xi u \, dY' \ \partial_Y \tilde{\omega}\right] = 0$$

$$\partial_\tau \varphi = v^\varphi$$

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$$\partial_\tau\varphi=v^\varphi$$

$$u-iv=\sum_n\int_{-\infty}^{\infty}\int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2}\frac{\tilde{\omega}(\xi',Y')}{\mathcal{K}^{\varepsilon}_{\varphi}(\xi-\xi',Y-Y')}J(\xi')d\xi' dY'$$

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$$\partial_\tau \varphi = v^\varphi$$

$$u - iv = \sum_n \int_{-\infty}^{\infty} \int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \frac{\tilde{\omega}(\xi', Y')}{\mathcal{K}^\varepsilon_\varphi(\xi-\xi', Y-Y')} J(\xi') d\xi' dY'$$

$$\mathcal{K}^\varepsilon_\varphi(\xi-\xi', Y-Y') = \frac{1}{\xi - \xi' + X(\xi) - X(\xi') + i\,[\varphi(\xi+X) - \varphi(\xi'+X') + \varepsilon(Y-Y')]}$$

with $J = 1 + \partial_\xi X(\xi,\tau)$.

The ”true” difficulty

The elliptic estimate $\|u\|_1^{(\alpha)} \leq c\|\tilde{\omega}\|_1^{(\alpha)}$ is not enough.
The problematic term is:

$$\int_0^Y \partial_\xi u \, dY' \, \partial_Y \tilde{\omega}$$

where a combination of the loss of one derivative in both terms (∂_ξ and ∂_Y) and of the linear growth (due to the integral of the velocity) makes it impossible to estimate this term.

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The hope is to separate the loss of the ξ derivative fom the linear growth, proving that the velocity, outside the layer, converges to some flow that we know *a priori* to be analytic.
A careful analysis of the asymptotic behavior of the velocity

- ouside the layer: $Y \leq O(\varepsilon^{-1})$
- at large distance from the layer $y \to \infty$. *Far field asymptotics*

is needed.

One can see that the velocity field can be written:

$$u - iv = \mathcal{BR}[\gamma, \phi] + \frac{1}{2}\left[\int_Y^{\infty} \tilde{\omega} dY' - \int_{-\infty}^{Y} \tilde{\omega} dY'\right] + \mathcal{R} =$$

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$$\mathcal{BR}[\gamma, \phi] = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \frac{\gamma(\xi')}{\mathcal{K}_{\phi}^{0}(\xi - \xi')} J(\xi') d\xi' \,,$$

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The remainder has a complicated expression

$$\mathcal{R}(\Omega, \phi) = \sum_n \int_{-\infty}^{\infty} \int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} [\Omega(\xi', Y') - \Omega(\xi, Y')] \left[\frac{1}{\mathcal{K}_{\phi}^{\varepsilon}} - \frac{1}{\mathcal{K}_{\phi}^{0}}\right] J(\xi') d\xi' dY'$$

However..

Approximating the velocity field inside the layer

Proposition (behavior of the velocity inside the layer, $Y \leq 1/\varepsilon$)

Let $\Omega \in B^3_{\rho,\theta,\mu}$ and $\phi \in B^3_{\rho}$. Then the following estimate holds:

$$\|\mathcal{R}(\Omega,\phi)(\cdot,Y)\|^{(\alpha)}_{1,\rho,\sigma} \leq c\varepsilon(1+Y)$$

The consequence is that, when $Y < 1/\varepsilon$:

$$u - iv = \mathcal{L}(\gamma,\varphi) + O(\varepsilon)$$

Notice however how the above requires high regularity for the vorticity

The far field approximation

Define the velocity generated by a vorticity concentrated on the curve ϕ with intensity γ.

$$u^f - iv^f \equiv \frac{1}{2\pi i} \sum_n \int_{-\infty}^{\infty} \int_{\xi+\pi(2n-1)/2}^{\xi+\pi(2n+1)/2} \frac{\gamma(\xi')}{\xi - \xi' + i\left[\phi - \phi' + \varepsilon Y\right]} d\xi'$$

Proposition (far field approximation)

Let $\Omega \in B^2_{\rho,\sigma.\mu}$, $\phi \in B^{2,\alpha}_{\rho}$ with $\|\phi\|^{(\alpha)}_{2,\rho} < 1/4$, $u \in B^2_{\rho,\sigma}$ with $\|u\|_{2,\rho,\sigma} < \Gamma$ and τ such that $\tau\Gamma < 1/5$. Moreover let $|Y| > \varepsilon^{-1}$. Then:

$$\|u + iv - (u^f + iv^f)\|^{(\alpha)}_{1,\rho,\sigma} \leq c \left[\left(\frac{1}{|Y|} + f(Y) \right) + O(e^{-\mu/(2\varepsilon)}) \right]$$

where $f(Y) \geq 0$ has a rate of decay in Y rapid enough to make it integrable in Y.

The asymptotic procedure

The fact that the estimate on $\mathcal{R}$ requires higher regularity does not allow to solve the Euler equation all at once, but requires an asymptotic procedure:

$$\tilde{\omega} = \omega_0 + \varepsilon\omega_1 \qquad \varphi = \varphi_0 + \varepsilon\varphi_1 \qquad u - iv = u_0 - iv_0 + \varepsilon\,(u_1 - iv_1)$$

where

$$u_0 - iv_0 = \mathcal{L}(\omega_0, \varphi_0)$$

and $\mathcal{R}(\omega_0, \varphi_0)$, although generated by ω_0, is included in the correction $\varepsilon\,(u_1 - iv_1)$.

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The equation to zeroth order

$$\partial_\tau \omega_0 + \frac{1}{1 + \partial_\xi X_0} \left\{ \left[\mathcal{L}^u(\omega_0, \varphi_0) - \mathcal{L}^u(\omega_0, \varphi_0)|_{Y=0}\right] \partial_\xi \omega_0 - \int_0^Y \partial_\xi \mathcal{L}^u(\omega_0, \varphi_0) dY' \partial_Y \omega_0 \right\} = 0$$

$$\partial_\tau \varphi_0 = \mathcal{L}^v\big|_{Y=0}$$

Recall that:

$$\mathcal{L}(\omega_0, \varphi_0) = \mathcal{BR}[\gamma_0, \varphi_0] + \frac{1}{2}\left[\int_Y^\infty \omega_0 dY' - \int_{-\infty}^Y \omega_0 dY'\right]$$

Threfore, to bound the hard term we need *a priori* bounds on γ_0 and φ_0.

Recall that:

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Threfore, to bound the hard term we need *a priori* bounds on γ_0 and φ_0.
Take

$$\partial_\tau \omega_0 + \frac{1}{1+\partial_\xi X_0}\left\{\left[\mathcal{L}^u(\omega_0, \varphi_0) - \mathcal{L}^u(\omega_0, \varphi_0)|_{Y=0}\right]\partial_\xi \omega_0 - \int_0^Y \partial_\xi \mathcal{L}^u(\omega_0, \varphi_0) dY' \partial_Y \omega_0\right\} = 0$$

and integrate in Y in $]-\infty, \infty[$.

$$\partial_\tau \gamma_0 - \mathcal{L}^u(\gamma_0, \varphi_0)|_{Y=0} \frac{\partial_\xi \gamma_0}{1+\partial_\xi X_0} + \frac{\partial_\xi\left(\gamma_0 \mathcal{BR}^u[\gamma_0, \varphi_0]\right)}{1+\partial_\xi X_0} = 0\,;$$

$$\partial_\tau\left(\gamma_0^+ - \gamma_0^-\right) + \frac{1}{1+\partial_\xi X_0}\left\{\frac{1}{2}\gamma_0 \partial_\xi \gamma_0 \tilde{\boldsymbol{t}}^u + \left(\gamma_0^+ - \gamma_0^-\right)\partial_\xi \mathcal{BR}[\gamma_0, \varphi_0]\right\} = 0$$

The strategy is therefore:

- Solve the above system to find γ_0 and φ_0. This givel the *skeleton* of the layer, i.e. the position φ of the curve and the vortex strength γ, but ignores how the vorticity is distributed.

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- Then solve the equation for ω_0 and find the $O(1)$ approximation of the vorticity distribution. This ω_0 is convected by the *local* velocity $\mathcal{L}$ only.
- Solve the equation for ω_1 and find the $O(\varepsilon)$ correction to the vorticity distribution, as well the correction φ_1.

The key ingredients: the Cauchy estimates

1. The Cauchy estimate for the *x*-derivative of an analytic function:

$$\|\partial_x \omega\|^{(\alpha)}_{1,\rho,\sigma.\mu} < \frac{\|\omega\|^{(\alpha)}_{1,\rho',\sigma.\mu,}}{\rho' - \rho} \qquad \rho' > \rho$$

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2. The Cauchy estimate for the ζ-derivative of an analytic function:

$$\|\chi(Y)\partial_Y \omega\|^{(\alpha)}_{1,\rho,\sigma.\mu} < \frac{\|\omega\|^{(\alpha)}_{1,\rho,\sigma',\mu}}{\sigma' - \sigma} \qquad \sigma' > \sigma$$

where $\chi(Y)$ is a bounded function such that $\chi(Y) = O(Y)$ when $Y \to 0$.

The key ingredients: a fixed point Theorem

3. ACK Theorem:
   A Banach scale is a collection of Banach spaces $\{B_\rho\}_{\rho \in I}$ such that:

$$\begin{array}{rcll} B_{\rho'} & \subseteq & B_\rho & \text{if} \quad \rho' > \rho \\ \|u\|_{\rho'} & \geq & \|u\|_\rho & \end{array}$$

   Suppose one has the differential problem in operator form:

$$\partial_t u = F(u,t) \qquad u(t=0) = u_0$$

Theorem (The Abstract Cauchy-Kowaleski Theorem)

Suppose the operator F is quasi-contractive:

$$\|F(u^1,t) - F(u^2,t)\|_{\rho'} \leq \frac{\|u^1 - u^2\|_\rho}{\rho' - \rho} \qquad \rho' > \rho$$

Then, if the initial datum $u_0 \in B_{\rho_0}$, then there exists a $\beta > 0$ such that:

$$u(t) \in B_{\rho_0 - \beta t}$$

Convergence

One has to show that the motion of the curve φ^ε converges, when the thickness $\varepsilon \to 0$ to the motion predicted by BR equations.

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One has to show that the motion of the curve φ^ε converges, when the thickness $\varepsilon \to 0$ to the motion predicted by BR equations.
Write the equation for the zero-th order vorticity intensity γ_0 and φ_0

$$\partial_\tau \gamma_0 - \mathcal{L}^u(\gamma_0, \varphi_0)|_{Y=0} \frac{\partial_\xi \gamma_0}{1 + \partial_\xi X_0} + \frac{\partial_\xi \left(\gamma_0 \mathcal{BR}^u[\gamma_0, \varphi_0]\right)}{1 + \partial_\xi X_0} = 0\,;$$

$$\partial_\tau \varphi_0 = \mathcal{L}^v|_{Y=0}$$

Then notice that

$$u^\varphi + iv^\varphi = \mathcal{L}_{|Y=0} + O(\varepsilon)$$

and recall that

$$\partial_t = \partial_\tau - \frac{1}{1 + X_\xi}\partial_\xi \qquad \partial_x = \frac{1}{1 + X_\xi}\partial_\xi$$

Translating these equations sback to the laboratory reference frame:

$$\partial_t \gamma_0 + \partial_x \left(\, BR^u \gamma_0\right) = O(\varepsilon)$$

$$\partial_t \varphi_0 + u^\varphi \partial_x \varphi_0 = v^\varphi$$

which are, up to $O(\varepsilon)$ the Birkhoff-Rott equations.

The result

Theorem

Suppose the initial datum is of vortex-layer type, i.e. with an $O(\varepsilon^{-1})$ vorticity concentrated in a small layer (size $O(\varepsilon)$) close to a curve. Suppose moreover that:

$$\|\tilde{\omega}^{in}\|^{(\alpha)}_{1,\rho,\sigma,\mu} < R \qquad \|\varphi^{in}\|^{(\alpha)}_{2,\rho} < 1/4$$

Then the vortex-layer structure is preserved for a time that does not depend on ε.
Moreover the curve moves, to the leading order, according to the Birlhoff-Rott equation.

Discussion

In *Existance de Nappes de Tourbillon en Dimension Deux*, *J.Am.Math.Soc.* 1991, Delort proved the following important result:

Delort, *J. Am. Math. Soc.* '91

If ω_0 is a bounded measure with positive singular part, then Euler equations (in the weak form) admit as solution a bounded measure ω_t, and $\boldsymbol{u} \in L^\infty_{loc}\left(\mathbb{R}, L^2_{loc}(\mathbb{R}^2, \mathbb{R}^2)\right)$.

However, despite of the title of the paper containing the words "Vortex sheet" he left unsolved the question whether his solution followed the Birkhoff-Rott equation (maybe imposing more regularity).

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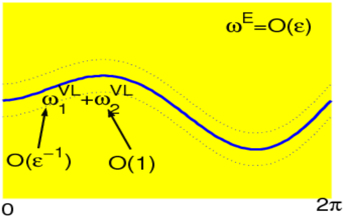
However, despite of the title of the paper containing the words ”Vortex sheet” he left unsolved the question whether his solution followed the Birkhoff-Rott equation (maybe imposing more regularity).

Our result clarifies that, if the data are analytic this is the case.

Discussion II

An important open problem is the justification of the VS model starting from the Navier-Stokes equation in the zero viscosity limit.

The conjecture is:



The solution of the Navier-Stokes solution, when the initial datum is of the vortex-layer type, should admit the following asymptotic expansion:

$$\boldsymbol{u}^{NS} = \begin{cases} \boldsymbol{u}_0^I + \sqrt{\nu}\boldsymbol{u}_1^I + O(\nu) & \text{close to } \varphi \\ \boldsymbol{u}_0^E + \sqrt{\nu}\boldsymbol{u}_1^E + O(\nu) & \text{away from } \varphi \end{cases}$$

while the curve φ moves according to

$$\partial_t \varphi = BR_0 + \sqrt{\nu} BR_1 + O(\nu)$$

Internal layer equations

In 2006 Caflisch and Sammartino derived, through a multiple scale expansion, the following equations that rule the flow inside the vortex layer:

Internal layer equations

$$
\begin{aligned}
&\partial_\tau u + \ddot{X} - 2\Omega \dot{Y} - \Omega^2 X - \dot{\Omega} Y + \\
&u \left[\partial_s u + \partial_s \dot{X} - Y \partial_s \Omega - \frac{\dot{Y}}{\rho} \right] + v \partial_N u + \partial_s p^L = \partial_{NN} u \\
&\partial_N p^L = 0 \\
&\partial_s u + \partial_N v = 0 \\
&u(s, N \to \pm\infty, t) \longrightarrow u^{\pm}(s, t)
\end{aligned}
$$

Well posedness: Caflisch and Sammartino 2006

Suppose we have an analytic curve $y = \phi_0(x)$ across which the velocity field has a rapid tangential variation $\gamma = U^+ - U^-$ with U^+ and U^- analytic and matching across the layer exponentially with a C^2 regularity. For a short time the equations ruling the fluid inside the layer (the internal layer equations) are well posed.

Correction to the BR equation

In 1994 Dhanak derived the following correction to the Birkhoff-Rott equation:

Dhanak and Moore's equations

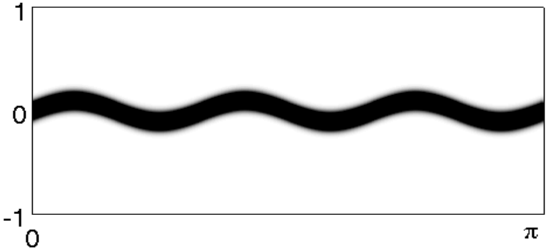
$$\frac{\partial Z^*}{\partial t} = -\frac{i}{2\pi} PV \int \frac{d\Gamma'}{Z(\Gamma,t) - Z(\Gamma',t)} - \sqrt{\nu} i \frac{\partial}{\partial \Gamma}\left[\delta_2 U^3 \frac{\partial Z^*}{\partial \Gamma}\right] + O(\nu)$$

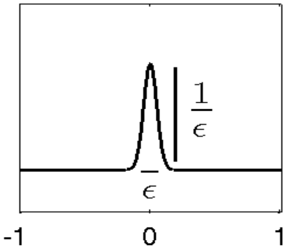
where U is the jump of the velocity and δ_2 is the *displacement thickness*:

$$\delta_2 = \int_{-\infty}^{\infty} \frac{(u^+ - u)(u^- - u)}{(u^- - u^-)^2} dn$$

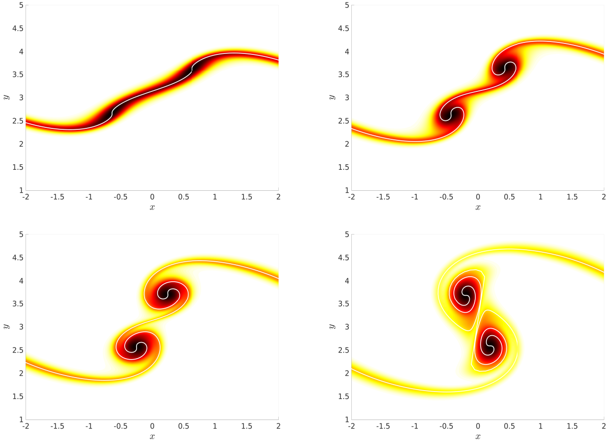
Dhanak's equation requires the knowledge of the flow inside the V-L which is given by the equation Caflisch and Sammartino derived.

Numerics I



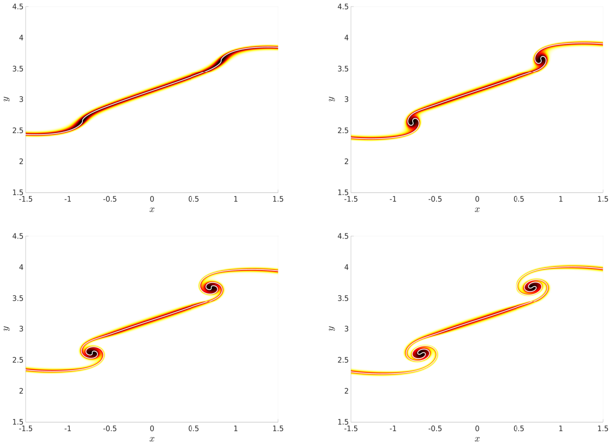


Numerics II



The vorticity distribution at various time for $\nu = 10^{-3}$. The white lines represent the material curve. The roll up behaviour typical of the vortex sheet motion is visible. Times $3.2, 4.6, 5.9$ are the time in which a new winding form in the material curve (the fourth winding forms at $t = 7.3$).

Numerics III



The vorticity distribution at various time for $\nu = 10^{-4}$. The white lines represents the material curve C

Prandtl singularity versus BR singularity

- Prandtl equations as well BR equation (try to) describe the limit $Re \to \infty$ of the NS solution.
- Both systems develop finite-time singulairity. (van Dommelen and Shen singularity, recently rigorously constructed by Kukavica, Vicol, Wang '15, see also E and Engquist '97)

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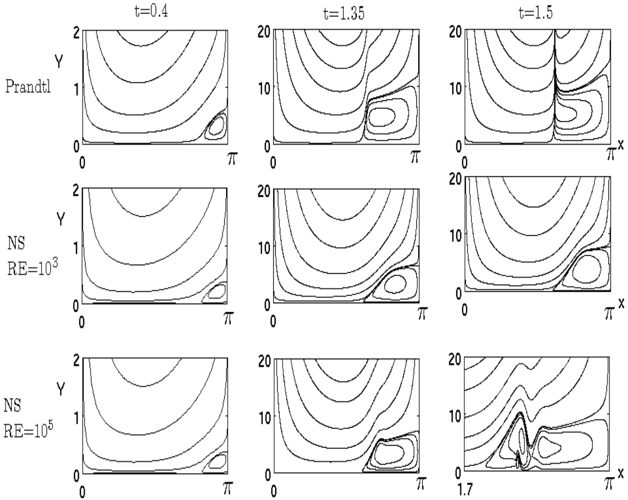
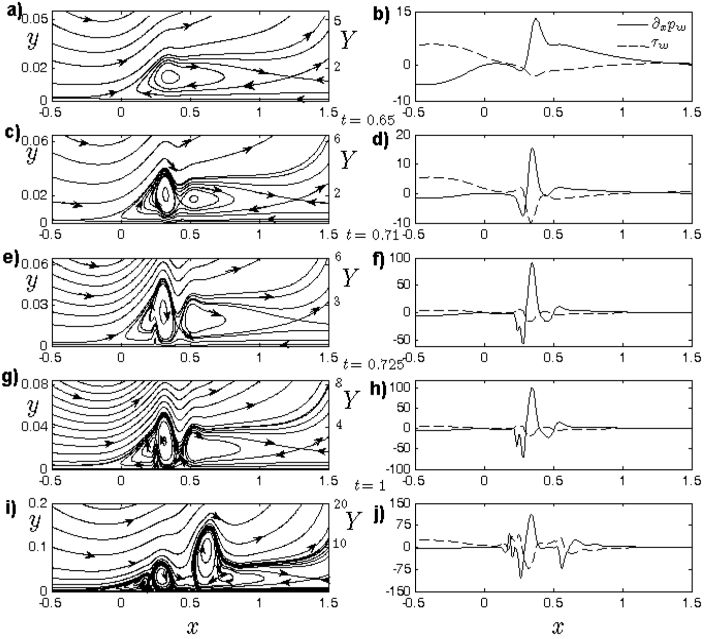
- Prandtl equations as well BR equation (try to) describe the limit $Re \to \infty$ of the NS solution.
- Both systems develop finite-time singulairity. (van Dommelen and Shen singularity, recently rigorously constructed by Kukavica, Vicol, Wang '15, see also E and Engquist '97)
- NS separation at the boundary: $O(1)$ vortical structures form, and the BL eventually detaches. $\longrightarrow$ Prandtl singularity ??
- NS solutions develop roll-up of the curve and an $O(1)$ vortical core forms. $\longrightarrow$ BR singularity ??

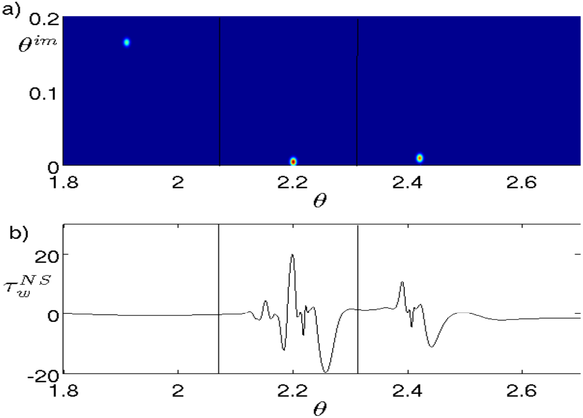
Streamlines for Prandtl and NS at different *Re* numbers.



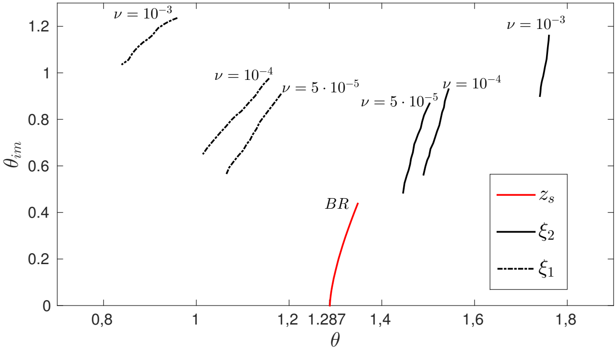

a)
b)
c)
d)
e)
f)
g)
h)
i)
j)
$\partial_x p_w$
τ_w
$t = 0.65$
$t = 0.71$
$t = 0.725$
$t = 1$
y
Y
x

NS complex singularities, the impulsively started disk

Is Prandtl singularity related to the above phenomena leading to separation?
Gargano, Sammartino, Sciacca and Cassel *J.Fluid.Mech* '14

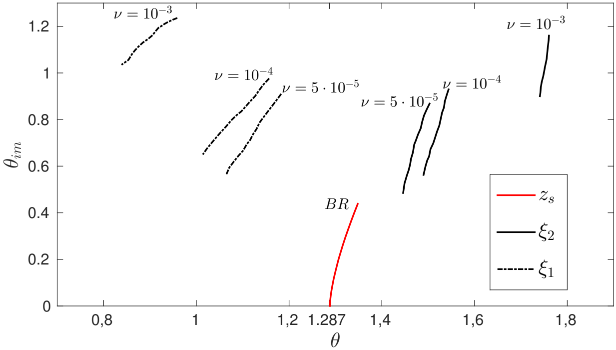


NS complex singularities, vortex layer motion



Two singularities are present. Both singularities seems to converge to the BR singularity. Caflisch, Gargano, Sammartino and Sciacca '15 *preprint*

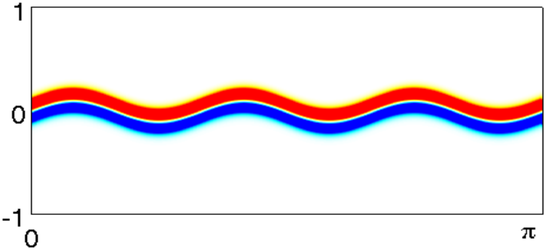
NS complex singularities, vortex layer motion

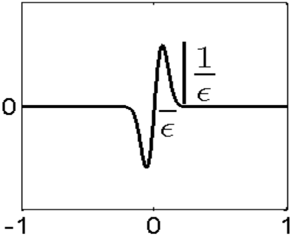


Two singularities are present. Both singularities seems to converge to the BR singularity. Caflisch, Gargano, Sammartino and Sciacca '15 *preprint*

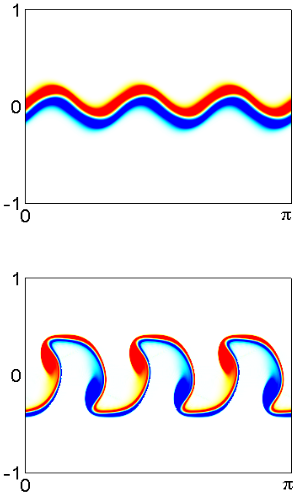
The Birkhoff-Rott singularity seems to be "the event" that causes the destruction of the layer structure and the formation of a thick vortex core.

Numerics III





Numerics IV



Thank you for your attention!