Dynamics of Ericksen-Leslie Model for Nematic Liquid Crystal Flows with General Leslie Stress

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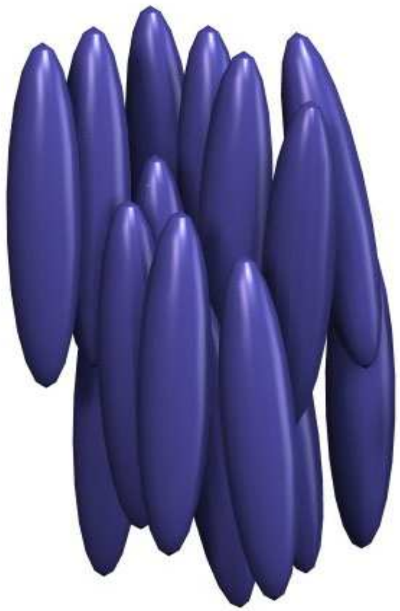
Liquid Crystals

Liquid Crystals

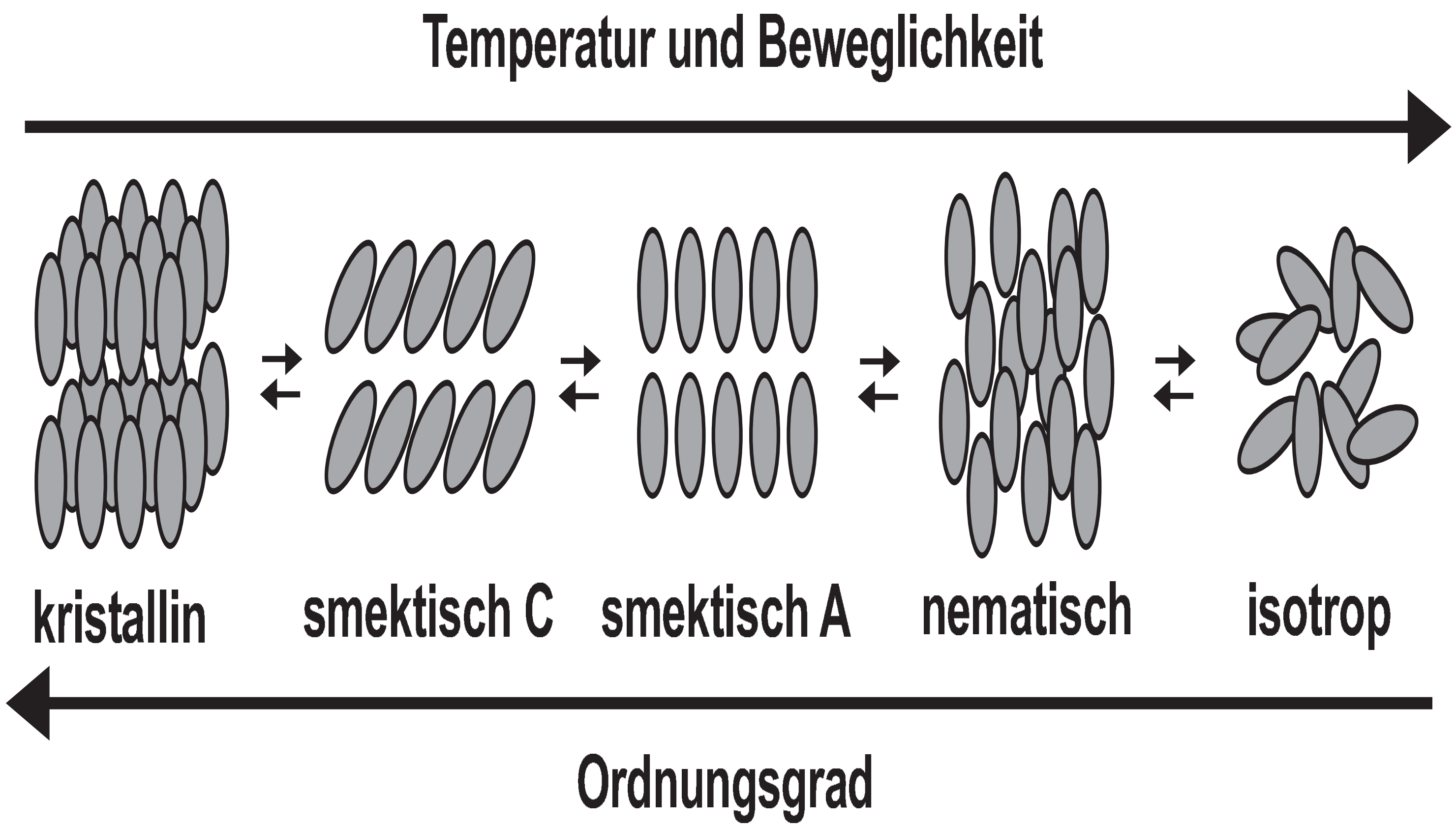
- material that has properties between those of conventional liquids and those of solids
- e.g. liquid crystals flow like liquid, but molecules are oriented in crystal like way
- many different phases characterized by optical properties and type of ordering
- main phases : nematic, smectic and cholesteric



- ordered, freely floating   layer structure   twisted structure

Nematic Liquid Crystals

Nematic versus crystal :



Nematic phase : molecules align along a direction

Some History

- 1888 : first discovery by chemists R. Reinitzer
- 1940 : synthesization of many liquid crystals
- 1933, 1958 : first continuum theory by Oseen and Frank for stationary case : find energy densities obeying constitutive laws, e.g. frame indifference (rigoroulsy, Virga 1994)
- 1949-86 : approach by Doi-Onsager
- 1962 : continuum theory for hydrodynamic flow by J. Ericksen
- 1968 : constitutive laws by F. Leslie
- 1991 : Nobel prize by P.-G. De Gennes, development of Q-tensor model
- 1995 : first rigorous analysis for simplified versions of Ericksen-Leslie model started by F. Lin and C. Liu
- 2013-2015 : well-posedness results for general Ericksen-Leslie model by Liu, Wu, Xu and Wang, P. Zhang, Z. Zhang and Li assuming various conditions on Leslie coefficients

The general Ericksen-Leslie Model in $\mathbb{R}^3$: original form

$$
\begin{aligned}
u_t + (u \cdot \nabla u) &= \text{div } \sigma && \text{on } (0,T) \times \Omega, \\
\text{div } u &= 0 && \text{on } (0,T) \times \Omega \\
d \times (g + \text{div}(\tfrac{\partial W}{\partial(\nabla d)}) - \tfrac{\partial W}{\partial d}) &= 0 && \text{on } (0,T) \times \Omega, \\
|d| &= 1 && \text{in } (0,T) \times \Omega \\
(u,d)(0) &= (u_0, d_0) && \text{in } \Omega
\end{aligned}
$$

- u velocity, σ stress tensor, d director describing orientation
- stress tensor $\sigma = -pI - \frac{\partial W}{\partial d_{ki}} d_{kj} + \sigma^{Leslie}$
- $W = W(d, \nabla d)$ Oseen-Frank energy functional given by $W = \frac{1}{2}[k_1(\text{div } d)^2 + k_2|d \times (\nabla \times d)|^2 + k_3|d(\nabla \times d)|^2 + (k_2 + k_4)(\text{tr } (\nabla d)^2 - (\text{div } d)^2)]$ with elasticity constants k_i
- $\sigma^{Leslie} = \alpha_1(dd : D)dd + \alpha_2 dN + \alpha_3 Nd + \alpha_4 D + \alpha_5 ddD + \alpha_6 Ddd$
- $D = D(u) = \frac{1}{2}[(\nabla u) + (\nabla u)^T]$
- $N = d_t + (u \cdot \nabla)d + V(u)d$ with $V(u) = \frac{1}{2}[(\nabla u) - (\nabla u)^T]$
- $g = \lambda_1 N + \lambda_2 Dd$

Aims

- strongly coupled, nonlinear system containing Navier-Stokes equations provided stress tensor would be Newtonian one
- understanding of model is not easy (at least for a mathematician)
- Aim I : Understanding of EL-model from physical principles also in non-isothermal situation : we use entropy principle
- Aim II : Understanding of EL-model from analytical point of view :
  a) local well-posedness in the strong sense, i.e. existence of a unique, local strong solution,
  b) determination of the set of all equilibria,
  c) global existence of a strong solution provided the intitial data are close to an equilibrium point in an appropriate norm,
  d) convergence of solutions to the equilibrium set,
  e) determination of the longtime behaviour of the solution.
- start with simplified situation
- general system reads as

General System

$$
\begin{cases}
\partial_t\rho + \mathrm{div}(\rho u) &= 0 \\
\rho(\partial_t + u\cdot\nabla)u + \nabla\pi &= \mathrm{div}\, S \\
\rho(\partial_t + u\cdot\nabla)\epsilon + \mathrm{div}\, q &= S : \nabla u - \pi \mathrm{div}\, u + \mathrm{div}(\rho\partial_{\nabla d}\psi\mathcal{D}_t d) \\
\gamma(\partial_t + u\cdot\nabla)d - \mu_V V d &= P_d\big(\mathrm{div}(\rho\frac{\partial\psi}{\partial\nabla d}) - \rho\nabla_d\psi\big) + \mu_D P_d D d, \\
\rho(0) = \rho_0, \quad u(0) = u_0, \quad \theta(0) &= \theta_0, \quad d(0) = d_0
\end{cases}
$$

in
in
in
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in

- boundary conditions : $u = 0, \quad q\cdot\nu = 0, \quad \nu_i\nabla_{\partial_i d}\psi d = 0 \quad$ on $\partial\Omega$
- thermodynamical laws
  $$\epsilon = \psi + \theta\eta, \quad \eta = -\partial_\theta\psi, \quad \kappa = \partial_\theta\epsilon = -\theta\partial_\theta\psi, \quad \pi = \rho^2\partial_\rho\psi,$$
  where $\psi = \psi(\rho,\theta,d,\nabla d)$ is density of free energy,
- constitutive laws

$$
\begin{cases}
S &= S_N + S_E + S_L^{stretch} + S_L^{diss}, \quad q = -\alpha_0\nabla\theta - \alpha_1(d|\nabla\theta)d. \\
S_N &= 2\mu_s D + \mu_b \mathrm{div}\, u\, I, \quad S_E = -\rho\frac{\partial\psi}{\partial\nabla d}[\nabla d]^\mathsf{T}, \\
S_L^{stretch} &= \frac{\mu_D+\mu_V}{2\gamma}\mathrm{n}\otimes d + \frac{\mu_D-\mu_V}{2\gamma} d\otimes\mathrm{n}, \quad \mathrm{n} = \mu_V V d + \mu_D P_d D d - \gamma\mathcal{D}_t d, \\
S_L^{diss} &= \frac{\mu_P}{\gamma}(\mathrm{n}\otimes d + d\otimes\mathrm{n}) + \frac{\gamma\mu_L+\mu_P^2}{2\gamma}(P_d D d\otimes d + d\otimes P_d D d) + \mu_0(Dd|d)d\otimes d
\end{cases}
$$

- $D = \frac{1}{2}(\nabla u + [\nabla u]^T)$, $V = \frac{1}{2}(\nabla u - [\nabla u]^\mathsf{T})$, $P = I - d\otimes d$.
- Oseen-Frank free energy density ψ given by
  $\psi^F = k_1(\mathrm{div}\, d)^2 + k_2|d\times(\nabla\times d)|_2^2 + k_3|d\cdot(\nabla\times d)|^2 + (k_2 + k_4)[\mathrm{tr}(\nabla d)^2 - (\mathrm{div}\, d)^2]$

The simplified Ericksen-Leslie model

For a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, consider

$$
\begin{aligned}
u_t - \Delta u + (u \cdot \nabla)u + \nabla \pi &= -\lambda \operatorname{div} \left([\nabla d]^T \nabla d\right) && \text{in } (0,T) \times \Omega, \\
d_t + (u \cdot \nabla) d) &= \gamma(\Delta d + |\nabla d|^2 d) && \text{in } (0,T) \times \Omega, \\
\operatorname{div} u &= 0 && \text{in } (0,T) \times \Omega, \\
|d| &= 1 && \text{in } (0,T) \times \Omega \\
(u, \partial_\nu d) &= (0,0) && \text{on } (0,T) \times \partial\Omega, \\
(u,d)(0) &= (u_0, d_0) && \text{in } \Omega
\end{aligned}
$$

where

- $u : (0,T) \times \Omega \to \mathbb{R}^n$: velocity
- $\pi : (0,T) \times \Omega \to \mathbb{R}$: pressure
- $d : (0,T) \times \Omega \to \mathbb{R}^n$: macroscopic molecular orientation

Approaches and Analysis since 1995

Above system has been considered rigorously first by

- Lin-Liu '95 : the term $|\nabla d|^2 d$ is replaced by $f(d) = \nabla F(d)$ for some F.
- in this case condition $|d| = 1$ cannot be preserved
- Lin, Lin-Liu : replace this condition by Ginzburg-Landau energy functional, i.e.
  $f(d) = \nabla F(d) = \nabla \frac{1}{4\varepsilon^2}(|d|^2 - 1)^2$.
  This yields Ginzburg-Landau approximating system

$$
\begin{aligned}
u_t - \nu \Delta u + (u \cdot \nabla)u + \nabla \pi &= -\lambda \mathrm{div}\, ([\nabla d]^T \nabla d) && \text{in } (0,T) \times \Omega,\\
d_t + (u \cdot \nabla) d) &= \gamma(\Delta d + \tfrac{1}{4\varepsilon^2}(1 - |d|^2)d && \text{in } (0,T) \times \Omega,\\
\mathrm{div}\ u &= 0 && \text{in } (0,T) \times \Omega,\\
(u, \partial_\nu d) &= (0,0) && \text{on } (0,T) \times \partial\Omega,\\
(u,d)(0) &= (u_0, d_0) && \text{in } \Omega
\end{aligned}
$$

Approaches

Two type of approaches :

- I Fluid-type approach : couple equation for d to methods for Navier-Stokes
- II Geometric approach by harmonic maps on spheres : couple fluid equation to this geometric approach

Results (very far from complete)

- I Lin, Lin-Liu '95 : f of Ginzburg-Landau type : global weak solutions for $d = 2, 3$, global strong solutions for $n = 2$
- II Lin, Wang : existence results via heat flow of harmonic maps
- I-II Wang '12 : $\Omega = \mathbb{R}^d$: global well-posedness provided data are small in $BMO^{-1} \times BMO$
- I Feireisl et al, '12 : weak solutions for non-isothermal situation
- I Hong, Li, Xin '14 : solutions of Ginzburg-Landau approximating system converge for $\varepsilon \to 0$ to original system

The Quasilinear Approach

Main idea : incorporate the term $\operatorname{div}([\nabla d]^T \nabla d)$ into the quasilinear operator A representing the left hand side of equation. More precisely, we rewrite

$$
\begin{aligned}
u_t - \nu \Delta u + (u \cdot \nabla) u + \nabla \pi &= -\lambda \operatorname{div}([\nabla d]^T \nabla d) && \text{in } (0,T) \times \Omega, \\
d_t + (u \cdot \nabla) d &= \gamma(\Delta d + |\nabla d|^2 d) && \text{in } (0,T) \times \Omega, \\
\operatorname{div} u &= 0 && \text{in } (0,T) \times \Omega, \\
(u, \partial_\nu d) &= (0,0) && \text{on } (0,T) \times \partial\Omega,
\end{aligned}
$$

as

$$
\partial_t \begin{pmatrix} u \\ d \end{pmatrix} + \begin{bmatrix} \mathcal{A}_q & \mathbb{P}\mathcal{B}_q(d) \\ 0 & \mathcal{D}_q \end{bmatrix} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} -\mathbb{P} u \cdot \nabla u \\ -u \cdot \nabla d + |\nabla d|^2 d \end{pmatrix}
$$

where

- $\mathcal{A}_q$ Stokes operator
- $\mathcal{D}_q$ Neumannn-Laplacian operator
- $\mathbb{P}$ Helmholtz projection
- $[\mathcal{B}_q(d)h]_i := \partial_i d_l \Delta h_l + \partial_k d_l \partial_k \partial_i h_l$
- thus : $\mathcal{B}_q(d)d = \operatorname{div}([\nabla d]^\mathsf{T} \nabla d)$

Liquid Crystals as Quasilinear Evolution Equation

We rewrite the (simplified) Ericksen-Leslie system as

$$\dot{z}(t) + A(z(t))z(t) = F(z(t)), \quad t \in J, \quad z(0) = z_0, \tag{1}$$

with

- state space $X_0 := L_{q,\sigma}(\Omega) \times L_q(\Omega)^n$, $1 < q < \infty$
- $\Omega \subset \mathbb{R}^d$ bounded domain with boundary $\partial\Omega \in C^2$
- the quasilinear part $A(z)$ given by the tri-diagonal matrix

$$A(z) = \begin{bmatrix} \mathcal{A}_q & \mathbb{P}\mathcal{B}_q(d) \\ 0 & \mathcal{D}_q \end{bmatrix},$$

- Stokes operator $\mathcal{A}_q = -\mathbb{P}\Delta$ in $L_{q,\sigma}(\Omega)$ with domain

$$D(\mathcal{A}_q) = \{u \in H^2_q(\Omega)^n : \operatorname{div} u = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega\}$$

- Neumann-Laplacian $\mathcal{D}_q$ in $L_q(\Omega)$ with domain

$$D(\mathcal{D}_q) := \{d \in H^2_q(\Omega)^n : \partial_\nu d = 0 \text{ on } \partial\Omega\}.$$

- $\mathcal{B}_q$ given by $[\mathcal{B}_q(d)h]_i := \partial_i d_l \Delta h_l + \partial_k d_l \partial_k \partial_i h_l$
- $F(z) = (-\mathbb{P}u \cdot \nabla u, -u \cdot \nabla d + |\nabla d|^2 d)$

Approach by maximal regularity

Local existence and regularity result for quasilinear problems

$$\dot{z}(t) + A(z(t))z(t) = F(z(t)), \quad t \in J, \quad z(0) = z_0,$$

- Let $X_1 \overset{d}{\hookrightarrow} X_0$ and $J = [0, a]$ for some $a > 0$
- Let $z_0 \in X_\gamma = (X_0, X_1)_{1-1/p,p}$ for $p \in (1, \infty)$
  (A) $A \in C^\omega(X_\gamma; \mathcal{L}(X_0, X_1))$ and $A(v)$ has maximal L_p-regularity for each $v \in X_\gamma$
  (F) $F \in C^\omega(X_\gamma; X_0)$.

Then, there exists $a > 0$, such that above system admits a unique solution z on $J = [0, a]$ in the regularity class

- $z \in H^1_p(J; X_0) \cap L_p(J; X_1) \hookrightarrow C(J; X_\gamma) \cap C((0, a]; X_\gamma)$
- the solution depends continuously on z_0 and can be extended to a maximal interval of existence $J(z_0) = [0, t^+(z_0))$.
- If z is such a solution on $J = [0, a]$, then

$$t^k[\frac{d}{dt}]^k z \in H^1_p(J; X_0) \cap L_p(J; X_1), \quad k \in \mathbb{N}.$$

- z is real analytic with values in X_1 on $(0, a)$.

Local Wellposedness

Summarizing, we obtain

- Let $2/p + n/q < 1$, $z_0 = (u_0, d_0) \in X_\gamma$. i.e. $u_0, d_0 \in B_{q,p}^{2-2/p}(\Omega)^n$ with $\operatorname{div} u_0 = 0$ in Ω
- Then there is a unique local solution $z \in H_p^1(J, X_0) \cap L_p(J; X_1)$ on J.
- Moreover,$z \in C([0,a]; X_\gamma) \cap C((0,a]; X_\gamma)$, i.e. the solution regularizes instantly in time.
- For each $k \in \mathbb{N}$, $t^k[\frac{d}{dt}]^k z \in H_p^1(J; X_0) \cap L_p(J; X_1)$ and $z \in C^\omega((0,a); X_1)$.

Condition $|d| = 1$ is preserved

Condition $|d| = 1$ is preserved by the flow induced by the Ericksen-Leslie model.

More precisely :

- Let $z \in H^1_p(J; X_0) \cap L_p(J; X_1)$ be a solution of Ericksen-Leslie model on $J = [0, a]$.
- Then $|d(t)| \equiv 1$ for all $t \in [0, a]$.
- Proof fairly easy : if $\varphi = |d|^2 - 1$, then

$$\partial_t |d|^2 = 2d \cdot \partial_t d, \quad \Delta |d|^2 = 2\Delta d \cdot d + 2|\nabla d|^2, \quad \nabla |d|^2 = 2d \cdot \nabla d,$$

  multiplication with d yields

$$\left\{ \begin{array}{rcll} \partial_t \varphi + u \cdot \nabla \varphi & = & \Delta \varphi + 2|\nabla d|^2 \varphi & \text{in } \Omega \\ \partial_\nu \varphi & = & 0 & \text{on } \partial\Omega, \\ \varphi(0) & = & 0 & \text{in } \Omega, \end{array} \right.$$

  provided $|d_0| \equiv 1$.
- Uniqueness of this parabolic convection-reaction diffusion equations yields $\varphi \equiv 0$, i.e. $|d| \equiv 1$.

Global Solutions

Consider the set of equilibria of (LCE) :

$$\mathcal{E} = \{z_* \in X_1 \colon A(z_*)z_* = F(z_*)\}.$$

and let A_0 be the linearizaton of (LCE). Assume

- (A) and (F) holds
- u_* is normally stable, i.e. 0 is semi-simple eigenvalue of A_0, i.e. $N(A_0) \oplus R(A_0) = X_0$ and $\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+$

Priniciple of Linearized Stability :
Then there exists $\rho > 0$ such that solution z with $z_0 \in B_{X_\gamma}(0, \rho)$ exists on $\mathbb{R}_+$ and converges exponentially to $u_\infty \in \mathcal{E}$ in X_γ as $t \to \infty$.

Dynamics of Solutions : Convergence to Equilibria

- $\mathcal{E}_0 = \{0\} \times \mathbb{R}^n$ is obviously an equilibria for (LCE)
- linearization of (LCE) at $z_* \in \mathcal{E}_0$ is given by $\dot{z} + A_* z = f, \quad z(0) = z_0$ in X_0, with $A_* = \mathrm{diag}(\mathcal{A}_q, \mathcal{D}_q), \quad D(A_*) = X_1$
- $u_* \in \mathcal{E}$ is normally stable, i.e. $\sigma(A_*) \setminus \{0\} \subset [\delta, \infty)$ for $\delta > 0$ and ker $(A_*) = \{0\} \times \mathbb{R}^n$

Theorem :
Let p, q as above. Then for each equilibrium $z_* \in \{0\} \times \mathbb{R}^n$ there exists $\epsilon > 0$ such that a solution $z(t)$ of (LCE) with initial data $z_0 \in X_\gamma$, $|z_0 - z_*|_{X_\gamma} \leq \epsilon$ exists globally and converges exponentially to $z_\infty \in \{0\} \times \mathbb{R}^n$ in X_γ, as $t \to \infty$

Lyapunov Functionals

- Define energy by $E := \frac{1}{2}\int_\Omega [|u|^2 + |\nabla d|^2] dx = E_{kin} + E_{pot}$
- Calculation yields

$$\frac{d}{dt}E(t) = -\int_\Omega [|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2] dx$$

  Hence, $E(t)$ is non-increasing along solutions
- E is even a strict Ljapunov functional, i.e. strictly decreasing along constant solutions.
- In fact : if $dE(t)/dt = 0$ at some time, then $\nabla u = 0$ and $\Delta d + |\nabla d|^2 d = 0$ in Ω. Hence $u = 0$ and d satisfies the nonlinear eigenvalue problem

$$\left\{ \begin{array}{rcll} \Delta d + |\nabla d|^2 d & = & 0 & \text{in } \Omega, \\ |d|^2 & = & 1 & \text{in } \Omega, \\ \partial_\nu d & = & 0 & \text{on } \partial\Omega. \end{array} \right. \tag{2}$$

Determination of Equilibria

- Lemma : if $d \in H_2^2(\Omega; \mathbb{R}^n)$ satisfies above eigenvalue problem, then d is constant in Ω.
- Proof : explicit calculation and induction by n
- Thus : energy functional E defined on X_γ is strict Ljapunov functional for (LCE). Equilibria are given by

$$\mathcal{E} = \{z_* = (u_*, d_*) : u_* = 0, \, d_* \in \mathbb{R}^n, \, |d_*| = 1\}$$

- Summary : rather complete understanding of dynamics of simplified model

Finite Time Blow Up for Dirichlet Boundary Conditions

Consider the case where $d = (0, 0, 1)$ on $\partial\Omega$, where $\Omega =$ open unit ball in $\mathbb{R}^3$.

Theorem (Huang, Lin, Liu, Wang, 2015)
a) There exists $\varepsilon_0 > 0$ such that if $u_0 \in C^\infty_{c,\sigma}(\Omega, \mathbb{R}^3)$ and $d_0 \in \{d \in C^\infty(\Omega, \mathbb{S}^2) : d = e \text{ on } \partial\Omega\}$ satisfies that d_0 is not homotopic to the constant map $e : \Omega \to \mathbb{S}^2$ relative to $\partial\Omega$ and

$$\int_\Omega (|u_0|^2 + |\nabla d_0|^2) \leq \varepsilon^2,$$

then short time smooth solution (u, π, d) subject to $d = e$ on $\partial\Omega$ blows up before $T = 1$.
b) There are examples of initial data (u_0, d_0) satisfying the above assumptions.

Back to Full Model

- how to understand the model and the many terms involved ?
- how to proceed with the analysis ?
- basic idea : try to understand the model from a thermodynamical point of view, develop a thermodynamically consistent extension of the model
- this understanding is also the key for analytical investigations

Balance Laws for Mass, Momentum and Energy

The balance laws for mass, momentum and energy read as

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho u) &= 0 && \text{in } \Omega, \\ \rho(\partial_t + u \cdot \nabla) u + \nabla \pi &= \operatorname{div} S && \text{in } \Omega, \\ \rho(\partial_t + u \cdot \nabla)\epsilon + \operatorname{div} q &= S : \nabla u - \pi \operatorname{div} u && \text{in } \Omega, \\ u = 0, \quad q \cdot \nu &= 0 && \text{on } \partial\Omega. \end{aligned}$$

- ρ density, u velocity, π pressure, ϵ internal energy, S extra stress and q heat flux.
- This gives conservation of the total energy since

$$\rho(\partial_t + u \cdot \nabla) e + \operatorname{div}(q + \pi u - Su) = 0 \quad \text{in } \Omega,$$

  with $e := |u|^2/2 + \epsilon$ energy density (kinetic and internal).
- Integrating over Ω yields

$$\partial_t \mathrm{E}(t) = 0, \quad \mathrm{E}(t) = \mathrm{E}_{kin}(t) + \mathrm{E}_{int}(t) = \int_\Omega \rho(t,x) e(t,x) dx,$$

  provided $q \cdot \nu = u = 0 \quad \text{on } \partial\Omega$

Basic Laws from Thermodynamics

- Ansatz : free energy $\psi = \psi(\rho, \theta, \tau)$, τ to be specified later.
- Then

$$\epsilon = \psi + \theta\eta \quad \text{internal energy},$$

$$\eta = -\partial_\theta \psi \quad \text{entropy},$$

$$\kappa = \partial_\theta \epsilon = -\theta \partial_\theta^2 \psi \quad \text{heat capacity}.$$

- classical case, Clausius-Duhem equation reads as

$$\rho(\partial_t + u \cdot \nabla)\eta + \mathrm{div}(q/\theta) = S : \nabla u/\theta - q \cdot \nabla\theta/\theta^2 + (\rho^2\partial_\rho - \pi)(\mathrm{div}\, u)/\theta \quad \text{in } \Omega.$$

- Hence, entropy flux Φ_η is given by $\Phi_\eta := q/\theta$
- entropy production by

$$\theta r := S : \nabla u - q \cdot \nabla\theta/\theta + (\rho^2 \partial_\rho - \pi)(\mathrm{div}\, u)$$

- boundary conditions employed yield that for total entropy N we have

$$\partial_t \mathrm{N}(t) = \int_\Omega r(t,x)dx \geq 0, \quad \mathrm{N}(t) = \int_\Omega \rho(t,x)\eta(t,x)dx,$$

  provided $r \geq 0$ in Ω.
- $\mathrm{div}\, u$ has no sign, hence $\pi = \rho^2 \partial_\rho \psi$, Maxwell's relation.
- this leads to $S : \nabla u \geq 0$ and $q \cdot \nabla\theta \leq 0$.

Summary

- Summarizing : conservation of energy and total entropy is non-decreasing provided these conditions, Maxwell and (BC) are satisfied
- Thus, these conditions ensure thermodynamical consistency of the model.
- example : classical laws due to Newton and Fourier :

$$S := S_N := 2\mu_s D + \mu_b \mathrm{div}\, u\, I, \quad 2D = (\nabla u + [\nabla u]^\top), \quad q = -\alpha_0 \nabla\theta.$$

Liquid Crystals

- $\psi = \psi(\rho, \theta, \tau)$ with $\tau = \frac{1}{2}|\nabla d|_2^2$
- d orientation vector or director satisfying $|d|^2 = 1$
- energy flux is now given by

$$\Phi_e := q + \pi u - Su - \Pi \mathcal{D}_t d, \quad \mathcal{D}_t = \partial_t + u \cdot \nabla d,$$

  where Π has to be modeled.
- constitutive laws

$$S = S_N + S_E + S_L, \quad S_E = -\theta \lambda \nabla d [\nabla d]^\mathsf{T}, \quad q = -\alpha_0 \nabla \theta - \alpha_1 (d \cdot \nabla \theta) d.$$

- S_N means Newton stress, S_E the Ericksen stress and S_L the Leslie stress
- the balance of entropy, i.e. the Clausius-Duhem equation reads as

$$\rho(\partial_t + u \cdot \nabla)\eta + \operatorname{div} \Phi_\eta = r,$$

  with $\Phi_\eta = q/\theta$ and

Evolution of director d

$$\begin{aligned}\theta r = &- q\cdot\nabla\theta/\theta + 2\mu_s|D|_2^2 + \mu_b|\mathrm{div}\, u|^2 + (\rho^2\partial_\rho\psi - \pi)\mathrm{div}\, u\\ &+ (\rho\partial_\tau\psi - \lambda)\nabla d[\nabla d]^{\mathsf{T}} : \nabla u + (\Pi - \rho\partial_\tau\psi\nabla d) : \nabla\mathcal{D}_t d\\ &+ S_L : \nabla u + (\mathrm{div}\Pi + \beta d)\cdot\mathcal{D}_t d.\end{aligned}$$

for some scalar function β.

- entropy production r nonnegative provided

$$\mu_s \geq 0, \quad 2\mu_s + n\mu_b \geq 0, \quad \alpha_0 \geq 0, \quad \alpha_0 + \alpha_1 \geq 0.$$

- The next five blue terms r have no sign, hence we require

$$\pi = \rho^2\partial_\rho\psi, \quad \lambda = \rho\partial_\tau\psi/\theta, \quad \Pi = \rho\partial_\tau\psi\nabla d$$

- next, assume Leslie stress S_L vanishes :
- $\gamma\mathcal{D}_t d = \mathrm{div}[(\rho\partial_\tau\psi)\nabla]d + \beta d$ for some $\gamma = \gamma(\rho,\theta,\tau) \geq 0$
- condition $|d|_2 = 1$ requires $\beta = \lambda|\nabla d|^2$
- this leads to the equation for d

$$\gamma(\partial_t + u\cdot\nabla)d = \mathrm{div}[\lambda\nabla]d + \lambda|\nabla d|^2 d,$$

- basic equation for evolution of the director field d
- entropy production : $\theta r = -q\cdot\nabla\theta/\theta + 2\mu_s|D|_2^2 + \mu_b|\mathrm{div}\, u|^2 + \frac{1}{\gamma}|\mathrm{a}|_2^2$, where $\mathrm{a} = \mathrm{div}[\lambda\nabla]d + \lambda|\nabla d|_2^2 d$

Stretching and Vorticity

introduce stretching stress : set $2V = \nabla u - [\nabla u]^{\mathsf{T}}$

- set $\mathrm{n} = \mu_V Vd + \mu_D P_d Dd - \gamma \mathcal{D}_t d$, where μ_V, μ_D, γ scalar functions of ρ, θ, τ, $\gamma > 0$
- define stretch tensor

$$S_L^{stretch} = \frac{\mu_D + \mu_V}{2\gamma} \mathrm{n} \otimes d + \frac{\mu_D - \mu_V}{2\gamma} d \otimes \mathrm{n}.$$

- entropy production becomes

$$S_L^{stretch} : \nabla u + \mathcal{D}_t d \cdot \mathrm{a} = \frac{1}{\gamma}(|\mathrm{a}|_2^2 + (\mathrm{n} + \mathrm{a}) \cdot (\mu_V Vd + \mu_D P_d Dd - \mathrm{a})).$$

- set $\mathrm{n} + \mathrm{a} = 0$, which yields equation for d including stretch

$$\gamma(\partial_t d + u \cdot \nabla d) = \mathrm{div}(\lambda \nabla) d + \lambda |\nabla d|_2^2 d + \mu_V Vd + \mu_D P_d Dd.$$

- it preserves the constraint $|d|_2 = 1$
- $-\mathrm{N}$, where N is entropy, is strict Lyapunov functional as soon as $\mu_s > 0, \quad 2\mu_s + n\mu_b > 0, \quad \alpha_0 > 0, \quad \alpha_0 + \alpha_1 > 0, \quad \gamma > 0$

Additional Dissipation

add additional dissipative terms in the stress tensor of the form

$$S_L^{diss} = \frac{\mu_P}{\gamma}(\mathrm{n}\otimes d + d\otimes \mathrm{n}) + \frac{\gamma\mu_L + \mu_P^2}{2\gamma}(P_d Dd\otimes d + d\otimes P_d Dd) + \mu_0(Dd|d)d\otimes d,$$

- S_L^{diss} is symmetric
- adding these terms will be thermodynamically consistent provided entropy production ensures that the total entropy production remains nonnegative
- total entropy production becomes

$$\theta r = [\alpha_0|\nabla\theta|_2^2 + \alpha_1(d|\nabla\theta)^2]/\theta + 2\mu_s|D|_2^2 + \mu_b|\mathrm{div}\, u|^2 \\ + \frac{1}{\gamma}|P_d \mathrm{div}(\lambda\nabla)d - \mu_P P_d Dd|_2^2 + \mu_L|P_d Dd|_2^2 + \mu_0(Dd|d)^2.$$

- for thermodynamical consistency need only

$$\alpha_0, \alpha_0 + \alpha_1 \geq 0, \quad \mu_s, 2\mu_s + n\mu_b \geq 0, \quad \mu_0, \mu_L \geq 0, \quad \gamma > 0.$$

General Model : compressible fluid, isotropic elasticity

$$
\begin{cases}
\partial_t \rho + \mathrm{div}(\rho u) &= 0 & \text{in } \Omega, \\
\rho(\partial_t + u\cdot\nabla)u + \nabla\pi &= \mathrm{div}\, S & \text{in } \Omega, \\
\rho(\partial_t + u\cdot\nabla)\epsilon + \mathrm{div}\, q &= S : \nabla u - \pi \mathrm{div}\, u + \mathrm{div}(\lambda \nabla d \mathcal{D}_t d) & \text{in } \Omega, \\
\gamma(\partial_t + u\cdot\nabla)d - \mu_V V d &= \mathrm{div}[\lambda\nabla]d + \lambda|\nabla d|^2 d + \mu_D P_d D d, & \text{in } \Omega, \\
\rho(0) = \rho_0, \quad u(0) = u_0, \quad \theta(0) &= \theta_0, \quad d(0) = d_0 & \text{in } \Omega.
\end{cases}
$$

- boundary conditions : $u = 0, \quad q\cdot\nu = 0, \quad \nu_i \nabla_{\partial_i d}\psi d = 0$ on $\partial\Omega$
- thermodynamical laws

$$\epsilon = \psi + \theta\eta, \quad \eta = -\partial_\theta \psi, \quad \kappa = \partial_\theta \epsilon = -\theta\partial_\theta\psi, \quad \pi = \rho^2 \partial_\rho \psi,$$

  where $\psi = \psi(\rho, \theta, \tau)$ with $\tau = \frac{1}{2}|\nabla d|^2$ density of free energy,
- constitutive laws

$$
\begin{cases}
S &= S_N + S_E + S_L^{stretch} + S_L^{diss}, \quad q = -\alpha_0 \nabla\theta - \alpha_1 (d|\nabla\theta)d. \\
S_N &= 2\mu_s D + \mu_b \mathrm{div}\, u\, I, \quad S_E = -\lambda \nabla d [\nabla d]^T, \\
S_L^{stretch} &= \frac{\mu_D + \mu_V}{2\gamma} \mathrm{n} \otimes d + \frac{\mu_D - \mu_V}{2\gamma} d \otimes \mathrm{n}, \quad \mathrm{n} = \mu_V V d + \mu_D P_d D d - \gamma \mathcal{D}_t d, \\
S_L^{diss} &= \frac{\mu_P}{\gamma}(\mathrm{n}\otimes d + d \otimes \mathrm{n}) + \frac{\gamma\mu_L + \mu_P^2}{2\gamma}(P_d D d \otimes d + d \otimes P_d D d) + \mu_0 (Dd|d) d \otimes d
\end{cases}
$$

General Model : Non-Isotropic Elasticity

- free energy $\psi = \psi(\rho, \theta, d, \nabla d)$
- Ericksen stress tensor $S_E = -\rho \frac{\partial \psi}{\partial(\nabla d)} [\nabla d]^T$
- equation for d : $\gamma \mathcal{D}_t d = P_d a + \mu_V V d + \mu_D P_d D d$
- $a = \partial_i(\rho \nabla_{\partial_i d} \psi) - \rho \nabla_d \psi$

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) &= 0 & \text{in} \\ \rho(\partial_t + u \cdot \nabla) u + \nabla \pi &= \operatorname{div} S & \text{in} \\ \rho(\partial_t + u \cdot \nabla)\epsilon + \operatorname{div} q &= S : \nabla u - \pi \operatorname{div} u + \operatorname{div}(\rho \partial_{\nabla d} \psi \mathcal{D}_t d) & \text{in} \\ \gamma(\partial_t + u \cdot \nabla) d - \mu_V V d &= P_d\left(\operatorname{div}(\rho \frac{\partial \psi}{\partial \nabla d}) - \rho \nabla_d \psi\right) + \mu_D P_d D d, & \text{in} \\ \rho(0) = \rho_0, \quad u(0) = u_0, \quad \theta(0) &= \theta_0, \quad d(0) = d_0 & \text{in} \end{cases}$$

- boundary conditions, thermodynamical and constitutive laws as before
- $S = S_N + S_E + S_L^{stretch} + S_L^{diss}$ and $S_E = -\rho \frac{\partial \psi}{\partial(\nabla d)} [\nabla d]^T$.

Analysis : Case of Incompressible Fluids

case of incompressible fluids, isotropic elasticity : $\rho = const$, $\tau = \frac{1}{2}|\nabla d|^2$:

$$\begin{cases} \rho \mathcal{D}_t u + \nabla \pi & = \operatorname{div} S & \text{in } \Omega, \\ \operatorname{div} u & = 0 & \text{in } \Omega, \\ \rho \mathcal{D}_t \epsilon + \operatorname{div} q & = S : \nabla u + \operatorname{div}(\lambda \nabla d \mathcal{D}_t d) & \text{in } \Omega, \\ \gamma \mathcal{D}_t d - \mu_V V d - \operatorname{div}[\lambda \nabla] d & = \lambda |\nabla d|^2 d + \mu_D P_d D d & \text{in } \Omega, \\ u = 0, \quad q \cdot \nu = 0, \quad \partial_\nu d & = 0 & \text{on } \partial\Omega, \\ \rho(0) = \rho_0, \quad u(0) = u_0, \quad \theta(0) & = \theta_0, \quad d(0) = d_0 & \text{in } \Omega. \end{cases} \tag{3}$$

- thermodynamical laws as above
- constitutive laws for S as above
- convenient to write the equation for energy as an equation for the temperature θ :

$$\rho \kappa \mathcal{D}_t \theta + \operatorname{div} q = (S - (1 - \theta \partial_\theta \lambda / \lambda) S_E) : \nabla u + \operatorname{div}(\lambda \nabla) d \cdot \mathcal{D}_t d + (\theta \partial_\theta \lambda) \nabla d : \nabla \mathcal{D}_t d$$

- third order terms in d appear !
- Hence : mixed order system

Approach via Quasilinear Evolution Equations

- define setting for principal variable $v = (u, \theta, d)$
- $v \in X_0$ where ground space $X_0 := L_{q,\sigma}(\Omega) \times Y_0$ with $L_q(\Omega) \times H^1_q(\Omega)$ for $1 < p, q < \infty$
- regularity space
  $X_1 = \{v \in H^2_q(\Omega) \cap L_{q,\sigma}(\Omega) : u = 0 \text{ on } \partial\Omega\} \times Y_1$ with
  $Y_1 = \{(\theta, d) \in H^2_q(\Omega) \times H^3_q(\Omega) : \partial_\nu \theta = \partial_\nu d = 0 \text{ on } \partial\Omega\}$
- consider solutions within the class

$$E(J) := v \in H^1_p(J; X_0) \cap L_p(J; X_1),$$

  where $J = (0, a)$ with $0 < a \leq \infty$
- if $1 > 1/2 + (n+2)/2q$, then time-trace X_γ of $E(J)$ is

$$X_\gamma = \{v \in B^{2(1-1/p)}_{qp}(\Omega)^{2n} \cap X_0 : \ d \in B^{1+2(1-1/p)}_{qp}, \ u = \partial_\nu \theta = \partial_\nu d = 0 \text{ on } \partial\Omega\}$$

- state manifold : $\mathcal{SM} = \{v \in X_\gamma : \ \theta(x) > 0, \ |d(x)|_2 = 1 \text{ in } \Omega\}$
- rewrite Ericksen-Leslie system as quasi-linear evolution equation in X_0 of the form

$$\dot{v} + A(v)v = F(v), \quad t > 0, \ v(0) = v_0,$$

Main Result : Incompressible Fluid, Isotropic Elasticity

- Let $J = (0, a)$, $1 < p, q < \infty$, $1 > 1/2 + 1/p + n/2q$
- let $\psi \in C^3$ and $\alpha, \mu_j, \gamma \in C^2$
- assume $\mu_s > 0, \alpha > 0, \mu_0, \mu_L \geq 0, \kappa, \gamma > 0, \lambda, \lambda + 2\tau\partial_\tau\lambda > 0$

Theorem :

- (Local Well-Posedness) :
  - Let $v_0 \in X_\gamma$. Then for some $a = a(v_0) > 0$, there is a unique solution
    $$v \in H^1_{p,\mu}(J, X_0) \cap L_{p,\mu}(J; X_1),$$
  - Moreover, $v \in C([0, a]; X_\gamma) \cap C((0, a]; X_\gamma)$, i.e. the solution regularizes instantly in time.
  - solution exists on a maximal time interval $J(v_0) = [0, t^+(v_0))$.
  - $|d(\cdot, \cdot)|_2 \equiv 1$, $\mathsf{E}(t) \equiv \mathsf{E}_0$, and $-N$ is a strict Lyapunov functional.
- (Stability of Equilibria) :
  Any equilibrium $v_* \in \mathcal{E}$ of above system is stable in X_γ in the sense that for each $v_* \in \mathcal{E}$ there is $\varepsilon > 0$ such that if $v_0 \in \mathcal{SM}$ with $|v_0 - v_*|_{X_{\gamma,\mu}} \leq \varepsilon$, then the solution v with initial value v_0 exists globally in time and converges at an exponential rate in X_γ to some $v_\infty \in \mathcal{E}$.

Key Ideas of Proof : Part I

Step 1 : Linearization :

- linearize system at initial value $v_0 = [u_0, \theta_0, d_0]^\mathsf{T}$ and drop all terms of lower order. This yields the principal linearization

$$\begin{cases} \mathcal{L}_\pi(\partial_t, \nabla) v_\pi &= f \quad \text{in } J \times \Omega, \\ u = \partial_\nu \theta = \partial_\nu d &= 0 \quad \text{on } J \times \partial\Omega, \\ u = \theta = d &= 0 \quad \text{on } \{0\} \times \Omega. \end{cases}$$

- here $v_\pi = [u, \pi, \theta, d]^\mathsf{T}$ unknown and $f = [f_u, f_\pi, f_\theta, f_d]^\mathsf{T}$ given data.
- differential operator $\mathcal{L}_\pi(\partial_t, \nabla)$ is defined via its symbol $\mathcal{L}_\pi(z, i\xi)$ given by

$$\mathcal{L}_\pi(z, i\xi) = \begin{bmatrix} M_u(z,\xi) & i\xi & 0 & izR_1(\xi)^\mathsf{T} \\ i\xi^\mathsf{T} & 0 & 0 & 0 \\ 0 & 0 & m_\theta(z,\xi) & -iz\theta_0 ba(\xi) \\ -iR_0(\xi) & 0 & -iba(\xi) & M_d(z,\xi) \end{bmatrix},$$

  with $b = \partial_\theta \lambda$, and $\lambda_1 = \partial_\tau \lambda$.
- parabolic part

$$\mathcal{L}(z, i\xi) = \begin{bmatrix} M_u(z,\xi) & 0 & izR_1(\xi)^\mathsf{T} \\ 0 & m_\theta(z,\xi) & iz\theta_0 ba(\xi) \\ -iR_0(\xi) & iba(\xi) & M_d(z,\xi) \end{bmatrix}. \qquad (4)$$

- entries of these matrices are given by

Symbols

$$
\begin{aligned}
m_\theta &= \rho\kappa z + \alpha|\xi|^2, \quad a(\xi) = \xi \cdot \nabla d_0 \\
M_d &= \gamma z + \lambda|\xi|^2 + \lambda_1 a(\xi) \otimes a(\xi) = m_d(z,\xi) + \lambda_1 a(\xi) \otimes a(\xi) \\
R_0 &= \frac{\mu_D + \mu_V}{2} P_0\xi \otimes d_0 + \frac{\mu_D - \mu_V}{2}(\xi|d_0)P_0 \\
R_1 &= (\frac{\mu_D + \mu_V}{2} + \mu_P)P_0\xi \otimes d_0 + (\frac{\mu_D - \mu_V}{2} + \mu_p)(\xi|d_0)P_0 \\
M_u &= \rho z + \mu_s|\xi|^2 + \mu_0(\xi|d_0)^2 d_0 \otimes d_0 + a_1(\xi|d_0)P_0\xi \otimes d_0 \\
&\quad + a_2(\xi|d_0)^2 P_0 + a_3|P_0\xi|^2 d_0 \otimes d_0 + a_4(\xi|d_0) d_0 \otimes P_0\xi.
\end{aligned}
$$

Here $P_0 = P_{d_0} = I - d_0 \otimes d_0$, and a_j are coefficients.

Part II : Maximal L^p-Regularity

Linearized system for $\mathcal{L}$ admits a unique solution $v_\pi = [u, \pi, \theta, d]^\mathsf{T}$ with

$$
\begin{aligned}
(u, \theta) &\in {}_0H^1_p(J; L_q(\Omega))^{n+1} \cap L_p(J; H^2_q(\Omega))^{n+1}, \\
\pi &\in L_p(J; \dot{H}^1_q(\Omega)), \\
d &\in {}_0H^1_p(J; H^1_q(\Omega))^n \cap L_p(J; H^3_q(\Omega))^n,
\end{aligned}
$$

if and only if

$$
(f_u, f_\theta) \in L_p(J; L_q(\Omega))^{n+1}, f_d \in L_p(J; H^1_q(\Omega))^n, f_\pi \in {}_0H^1_p(J; H^{-1}_q(\Omega)) \cap L_p(J; H^1_q(\Omega)).
$$

- to prove this, set $J = \mathrm{diag}(I, 1/\theta_0, zI)$
- show that symbol $\bar{J}\mathcal{L}$ is accretive for $\mathrm{Re}\, z > 0$, i.e. the associated system is strongly elliptic.
- note we do not need any structural conditions on coefficients
- How to deal with mixed order situation ?
- perform a Schur reduction to reduce to symbol depending only on u.
- resulting generalized Stokes symbol for (u, π) is strongly elliptic
- apply maximal regularity result for non-Newtonian fluids to obtain maximal regularity for $\mathbb{R}^n$.
- half space : verify Lopatinskii-Shapiroo condition
- domains : localization prodecure

The subsystem for $w := (\theta, d)$

Consider subsystem associated with $w := (\theta, d)$. The principal part of the linearization becomes

$$\begin{aligned} \partial_t w + \mathcal{A}(w_0, \nabla) w &= f && \text{in } \Omega, \\ \partial_\nu w &= 0 && \text{on } \partial\Omega, \\ w(0) &= w_0 && \text{in } \Omega. \end{aligned}$$

- where $\mathcal{A} = \mathcal{A}(w_0, \nabla)$ is given by

$$\mathcal{A} = \begin{bmatrix} -a_0\Delta - a_1 \nabla d_0^{\mathsf{T}} \nabla d_0 : \nabla^2, & -b_0 \nabla d_0 : (\lambda_0 \Delta + \partial_\tau \lambda_0 [\nabla d_0]^{\mathsf{T}} \nabla d_0 : \nabla^2) \nabla \\ b_1 [\nabla d_0]^{\mathsf{T}} \nabla, & -\gamma_0^{-1} (\lambda_0 \Delta + \partial_\tau \lambda_0 [\nabla d_0]^{\mathsf{T}} \otimes \nabla d_0 : \nabla^2). \end{bmatrix}.$$

- Here $\kappa_0 = \kappa(\theta_0, \tau_0)$ etc., and

$$a_0 = \frac{\alpha_0}{\rho \kappa_0}, \quad a_1 = \frac{[\partial_\tau \epsilon_0]^2}{\theta_0 \gamma_0 \kappa_0}, \quad b_0 = \frac{\partial_\tau \epsilon_0}{\gamma_0 \kappa_0}, \quad b_1 = \frac{\partial_\tau \epsilon_0}{\gamma_0 \theta_0}.$$

- $\mathcal{A}(w_0, \nabla)$: second order diagonal, but third and first order off-diagonal
- This is a mixed-order problem

Maximal Regularity for Subsystem in $Y_0 := L_q(\Omega) \times H^1_q(\Omega;\mathbb{R}^n)$

- reduced variable $w_{red} = [\theta, d_{red}]^{\mathsf{T}}$ where $d_{red} = c(\xi)\cdot d$ yields reduced symbol $\mathcal{A}_{red}(\xi)$

$$\mathcal{A}_{red}(\xi) = \begin{bmatrix} a_0|\xi|^2 + a_1|c(\xi)|^2 & -ib_0(\lambda_0|\xi|^2 + \partial_\tau\lambda_0|c(\xi)|^2) \\ ib_1|c(\xi)|^2 & \frac{\lambda_0}{\gamma_0}|\xi|^2 + \frac{\partial_\tau\lambda_0}{\gamma_0}|c(\xi)|^2 \end{bmatrix}.$$

- now : reduced symbol is homogeneous of second order and normally elliptic in the sense that $\sigma(\mathcal{A}_{red}(\xi)) \subset (0,\infty)$ for each $\xi \neq 0$.
- hence : reduced equation has maximal regularity
- regain d by solving

$$\partial_t d - \frac{\lambda_0}{\gamma_0}\Delta d = f_d^1 := f_d + i\frac{\partial_\tau\lambda_0}{\gamma_0}c(\nabla)d_{red} - b_1 c(\nabla)\theta, \quad t>0,\ d(0)=0,$$

  with $d \in {}_0H^1_p(J; H^1_q(\mathbb{R}^n;\mathbb{R}^n)) \cap L_p(J; H^3_q(\mathbb{R}^n;\mathbb{R}^n))$
  for $f_d^1 \in L_p(J; H^1_q(\mathbb{R}^n;\mathbb{R}^n))$.

Part III : Local Existence

Rewrite Ericksen-Leslie system as quasi-linear evolution equation

$$\dot{v} + A(v)v = F(v), \quad t > 0,\ v(0) = v_0,$$

- $v = (u, \theta, d)$ and Helmholtz projection P is applied to the equation for u
- base space $X_0 := L_{q,\sigma}(\Omega) \times Y_0$ with $Y_0 = L_q(\Omega) \times H^1_q(\Omega)$
- quasilinear theory : for some $a = a(z_0) > 0$, there is a unique solution

$$z \in H^1_p(J, X_0) \cap L_p(J; X_1), \quad J = [0, a],$$

  of EL-system on J.
- Moreover, $t[\frac{d}{dt}]z \in H^1_p(J; X_0) \cap L_p(J; X_1)$
- $|d(t,x)|_2 \equiv 1$, $\mathsf{E}(t) \equiv \mathsf{E}_0$, and $-N$ is a strict Lyapunov functional
- Ericksen-Leslie system generates a local semi-flow in its natural state manifold $\mathcal{SM}$.

Part IV : Dynamics

- Linearization of (EL)-System at an equilibrium $v_* = (0, \theta_*, d_*)$ is given by $A_* = A(v_*)$ in X_0.
- This operator has maximal L_p-regularity, it is the negative generator of a compact analytic C_0-semigroup, and it has *compact resolvent*.
- $\sigma(A_*)$ consists only of countably many eigenvalues of finite multiplicity, which have all positive real parts, hence are stable, except for 0.
- The eigenvalue 0 is semi-simple. Its eigenspace is given by

  $$\mathrm{N}(A_*) = \{(0, \vartheta, \mathrm{d}) : \vartheta \in \mathbb{R}, \mathrm{d} \in \mathbb{R}^n\},$$

  hence it coincides with the set of constant equilibria $\bar{\mathcal{E}}$
- apply generalized principle of linearized stability, to prove the stability assertion

Case of Compressible Fluids

Recall that compressible models reads as

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) &= 0 & \text{in } \Omega, \\ \rho(\partial_t + u \cdot \nabla) u + \nabla \pi &= \mathrm{div}\, S & \text{in } \Omega, \\ \rho(\partial_t + u \cdot \nabla)\epsilon + \mathrm{div}\, q &= S : \nabla u - \pi \mathrm{div}\, u + \mathrm{div}(\lambda \nabla d \mathcal{D}_t d) & \text{in } \Omega, \\ \gamma(\partial_t + u \cdot \nabla) d - \mu_V V d &= \mathrm{div}[\lambda \nabla] d + \lambda |\nabla d|^2 d + \mu_D P_d D d, & \text{in } \Omega, \\ \rho(0) = \rho_0, \quad u(0) = u_0, \quad \theta(0) &= \theta_0, \quad d(0) = d_0 & \text{in } \Omega. \end{cases}$$

with boundary conditions, thermodynamical and constitutive laws as above

- approach is now much more involved, due to the hyperbolic part of the system
- local well-posedness and also the stability part are proven by introducing Lagrangian coordinates
- in contrast to incompressible case, here we cannot use $d \in H^3_q$, as the density ϱ does not have enough regularity.
- solution space in now given by

$$\rho \in H^1_p(J; H^1_q(\Omega)), \; (u, \theta) \in H^1_p(J; L_q(\mathbb{R}^{n+1}) \cap L_p(J; H^2_q(\mathbb{R}^{n+1})),$$

  while the director lies in

$$d \in H^2_p(J; {}_0H^{-1}_q(\Omega; \mathbb{R}^n)) \cap H^{1/2}_p(J; H^2_q(\Omega; \mathbb{R}^n)) \hookrightarrow H^1_p(J; H^1_q(\Omega; \mathbb{R}^n)),$$

Compressible case : strong well-posedness

Let $v = (\varrho, u, \theta, d)$.

- state space $X_\gamma := H^1_q(\Omega) \times B^{2-2/p}_{qp}(\Omega; \mathbb{R}^{n+1}) \times H^2_q(\Omega; \mathbb{R}^n)$
- state manifold $\mathcal{SM} = \{v \in X_\gamma : \; \varrho, \theta > 0, \; |d|_2 = 1 \text{ in } \Omega, \; \mathrm{div}(\lambda\nabla)d \in B^{1-2/p}_{qp}(\Omega; \mathbb{R}^n) u = \alpha_0 \partial_\nu \theta + \alpha_1 (d|\nu)\partial_d \theta = \partial_\nu d = 0 \; \text{ on } \partial\Omega\}$
- manifold of equilibria :
  $\mathcal{E} = \{v_* = (\varrho_*, 0, \theta_*, d_*) \in \mathbb{R}^{2n+2} : \; \rho_*, \theta_* > 0, \; |d_*|_2 = 1\}$

Theorem

Regularity assumptions on coefficients as above. Then :

- compressible EL-system generates a local semi-flow in $\mathcal{SM}$, solution exists on a maximal time interval $[0, t_+(v_0))$
- total mass M and total energy E are constant and negative total entropy $-$N is a strict Lyapunov functional.
- any equilibrium $v_* \in \mathcal{E}$ of EL-system is stable in $\mathcal{SM}$
- for each $v_* \in \mathcal{E}$ there is $\varepsilon > 0$ such that if $v_0 \in \mathcal{SM}$ with $|v_0 - v_*|_{X_\gamma} \leq \varepsilon$, then the solution v of EL-system with initial value v_0 exists globally and converges at an exponential rate in $\mathcal{SM}$ to some $v_\infty \in \mathcal{E}$.

Thank you very much