Added mass effect in forward and inverse fluid-structure interaction algorithms

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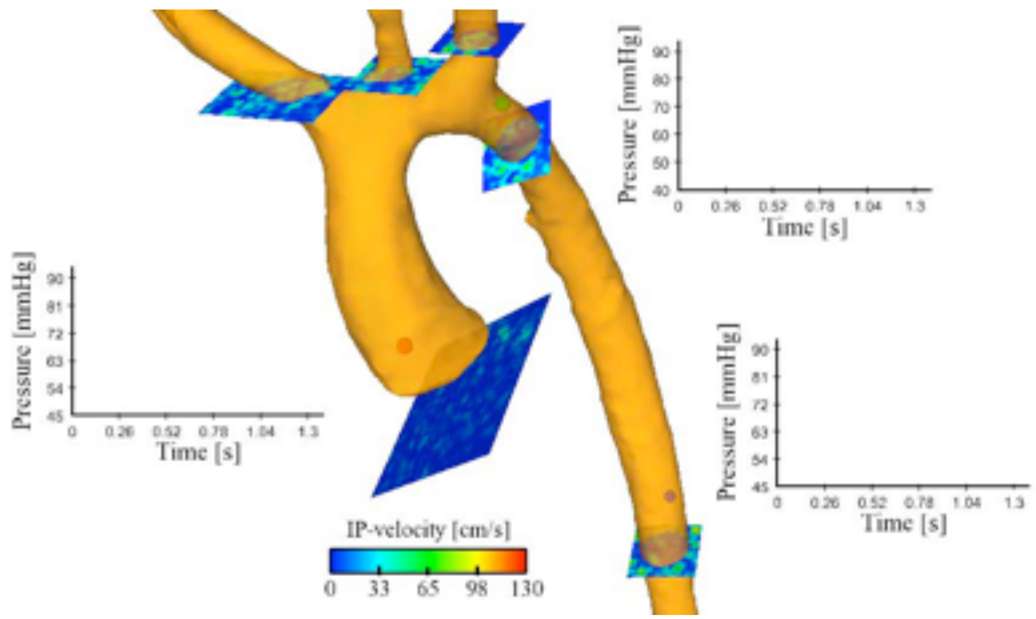
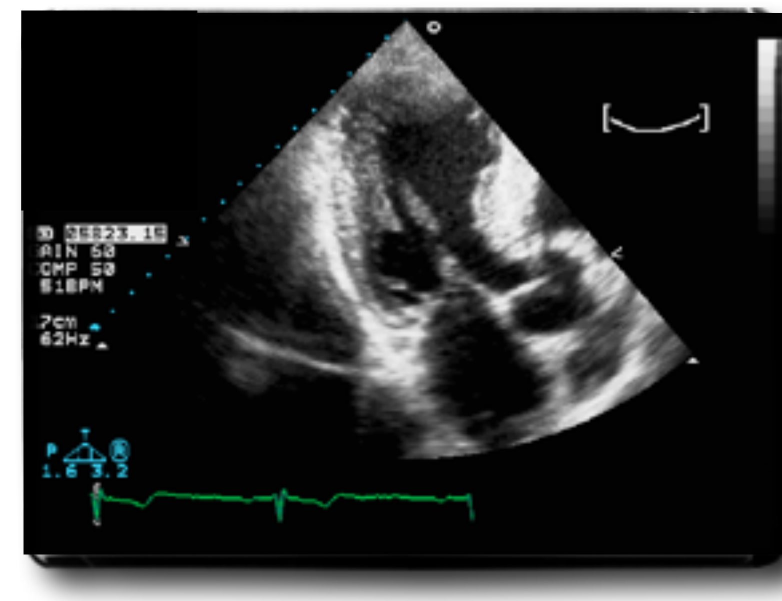
Fluid-structure interaction in blood flows



KCL (euHeart)



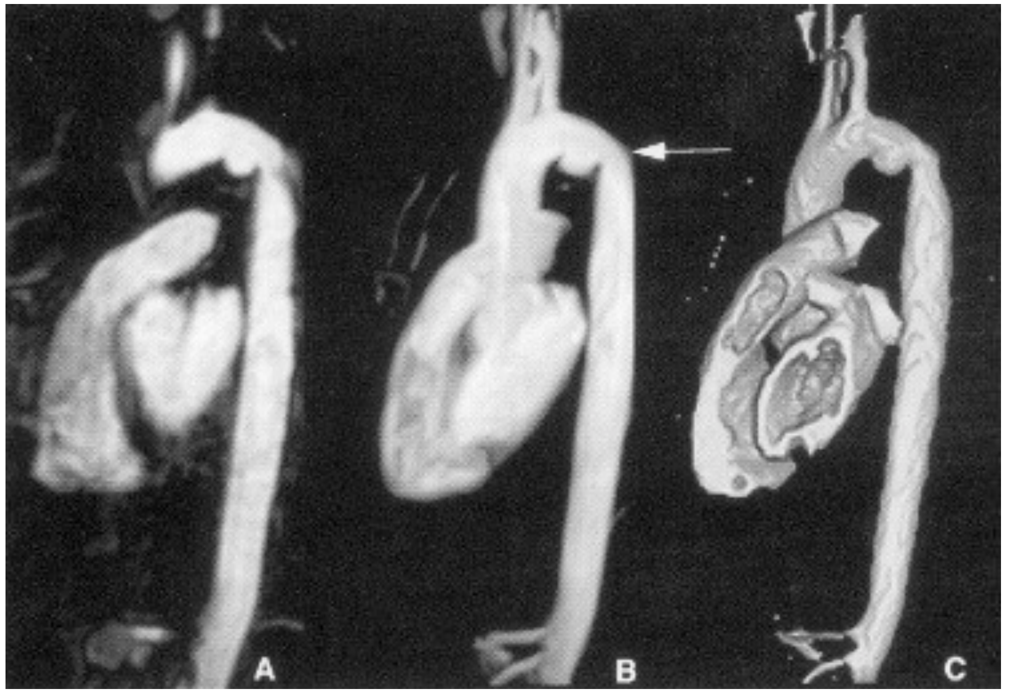
Hôpital Laval

$$\rho^{\mathrm{f}}\left(\frac{\partial \boldsymbol{u}}{\partial t}_{|\widehat{\boldsymbol{x}}} + (\boldsymbol{u}-\boldsymbol{w})\cdot\boldsymbol{\nabla}\boldsymbol{u}\right) - 2\mu\,\mathbf{div}\boldsymbol{\epsilon}(\boldsymbol{u}) + \boldsymbol{\nabla} p = \mathbf{0}, \quad \text{in} \quad \Omega^{\mathrm{f}}(t)$$

$$\operatorname{div}\boldsymbol{u} = 0, \quad \text{in} \quad \Omega^{\mathrm{f}}(t)$$

$$\rho^{\mathrm{s}}\frac{\partial^2 \boldsymbol{d}}{\partial t^2} - \mathbf{div}\big(\boldsymbol{F}(\boldsymbol{d})\boldsymbol{S}(\boldsymbol{d})\big) = \mathbf{0}, \quad \text{in} \quad \widehat{\Omega}^{\mathrm{s}}$$

Possible application: avoid clinical exams ?



Source: O. Peruta

- **Example:** aortic coarctation
- After surgical repair, patients must be followed on a regular basis
- Exercise test is often necessary to assess the patient condition
- **Question:** With computer simulations, can we extrapolate the rest test to avoid the stress test ?

- Maybe... if we are able to "personalize" an FSI model of the aorta

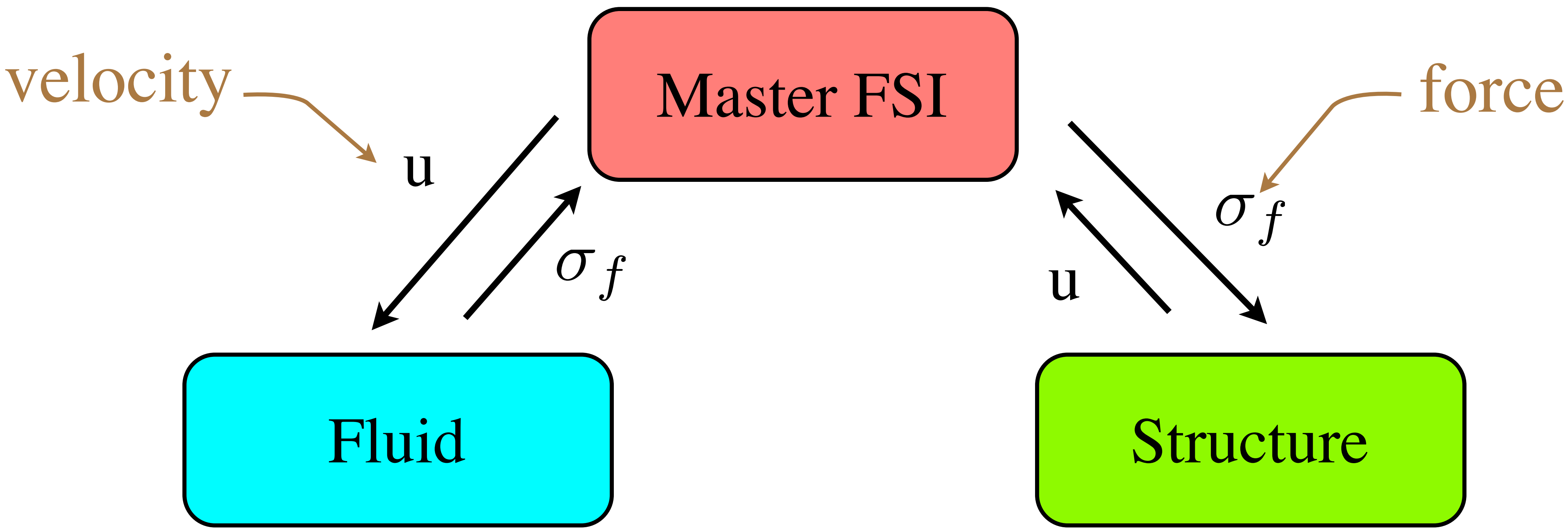
In collaboration with R. Hose (Sheffield), I. Valverde, P. Beerbaum (KCL)

Outline

- Forward problem in Fluid-Structure Interaction
- Inverse problem in Fluid-Structure Interaction

Fluid-Structure coupling

- **Partitioned approach:**



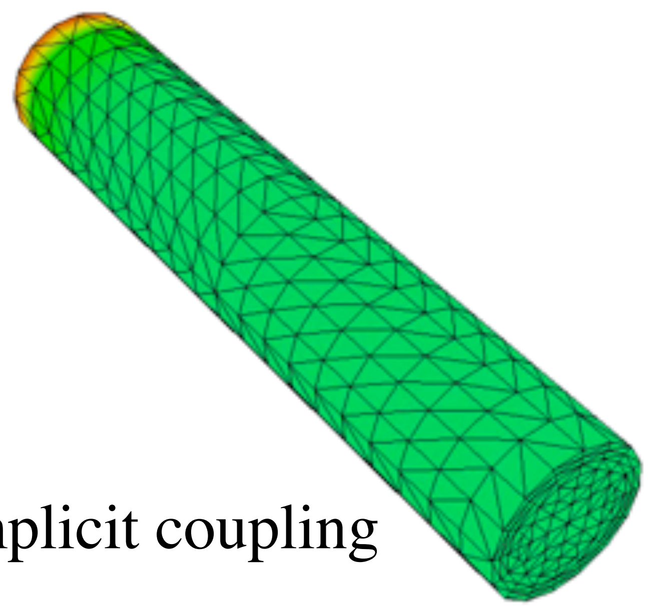
- **Explicit** scheme : one iteration Fluid/Structure at each time step
- **Implicit** scheme : many Fluid/Structure subiterations at each time step

Explicit coupling: some observations

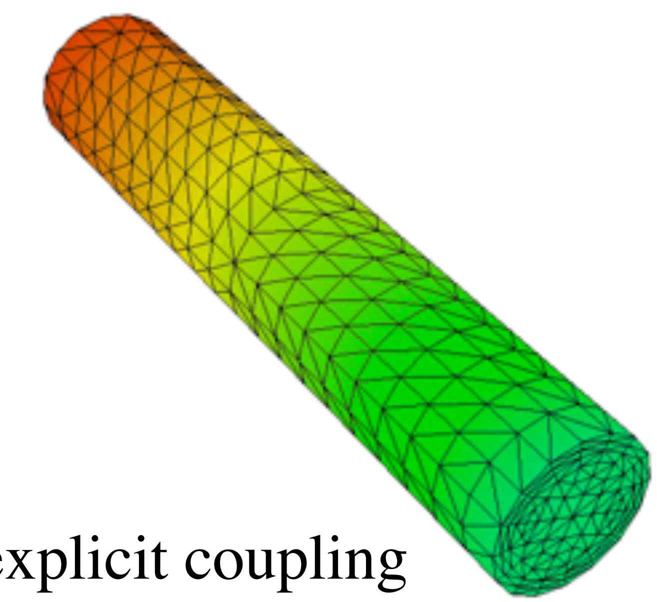
- Explicit algorithms are a priori very efficient:

$$\text{FSI cost} \approx \text{FLUID cost} + \text{SOLID cost}$$

- ... but naive Dirichlet-Neumann iterations are unstable !



implicit coupling



explicit coupling

- Explicit coupling is stable and widely used in aeroelasticity !
- Empirical observations for explicit coupling in blood flows:
  - ➡ Instabilities disappear when the solid density is (artificially) increased
  - ➡ Instabilities are independent of the time step
  - ➡ The instability is sensitive to the **length** of the domain

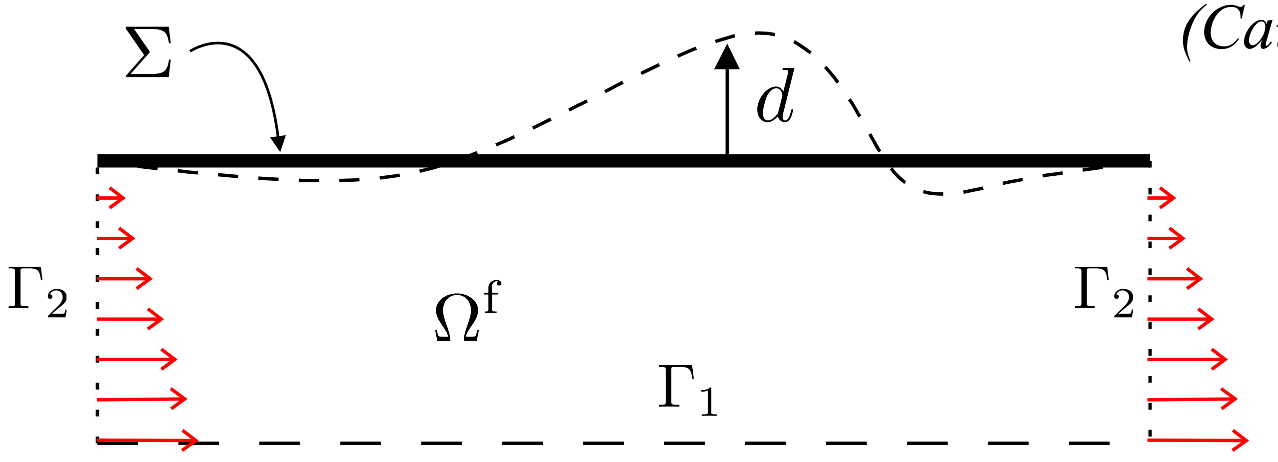
Implicit / Explicit coupling

Two approaches:

- Improve implicit iterations (Fixed point, Newton, ...)
  - Le Tallec-Mouro (1999) Wall-Ramm (2001), Fernández-Moubachir (2003), Matthies-Steindorf (2003), JFG-Vidrascu (2003), Mischler-van Brummelen-de Borst (2005), Deparis-Discacciati-Quarteroni (2005), Badia-Nobile-Vergara (2007), Vierendeels (2006), Vierendeels-Lanoye-Degroote-Verdonck (2007), Degroote-Annerel-Vierendeels (2010), and **many others**...
- Devise explicit coupling algorithms:
  - Projection semi-implicit coupling: Fernández-JFG-Grandmont (2007), Badia-Quaini-Quarteroni (2008)
  - Robin-Neuman : Burman-Fernández (2008)
  - Kinematically coupled time-splitting: Glowinski-Cavallini-Canic (2009), Fernández (2012)

A 2D simplified model

(Causin, JFG, Nobile, 2005)



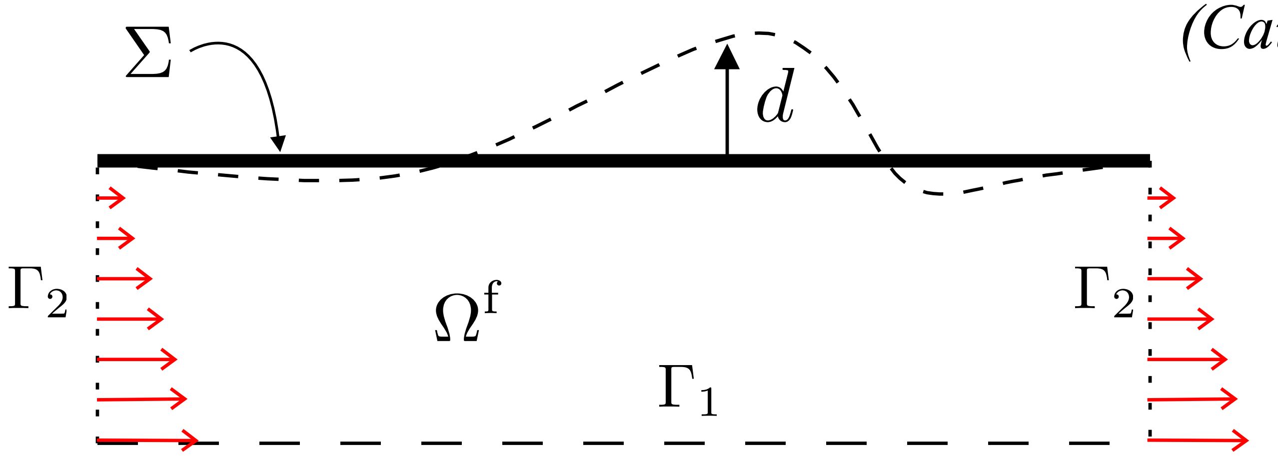
- Solid: string model (small displacements)

$$\rho^{s}\varepsilon\ddot{d} + Ld = p_{|\Sigma}, \quad \text{in} \quad \Sigma,$$

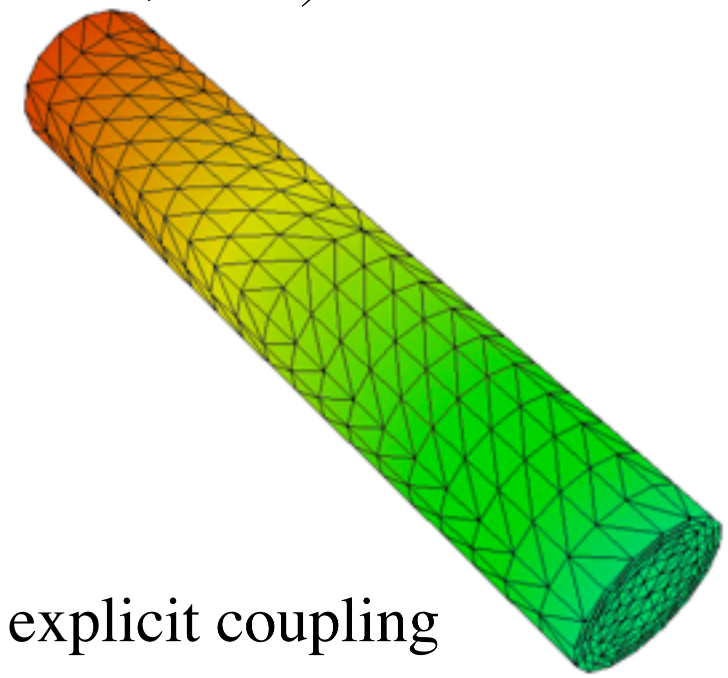
with

- d: vertical displacement
- ε: vessel thickness
- L: linear operator (for instance $L\eta = a\eta - b\frac{\partial^2\eta}{\partial x^2}$)

A 2D simplified model



(Causin, JFG, Nobile, 2005)



explicit coupling

- Solid: string model (infinitesimal displacements)

$$\rho^{\mathrm{s}} \varepsilon \ddot{d} + L d = p_{|\Sigma}, \quad \text{in} \quad \Sigma,$$

- Fluid: fixed fluid domain, no viscous/convective terms

$$\left\{\begin{aligned} \rho^{\mathrm{f}} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} p &= 0, && \text{in} \quad \Omega^{\mathrm{f}} \\ \operatorname{div} \boldsymbol{u} &= 0, && \text{in} \quad \Omega^{\mathrm{f}} \\ \boldsymbol{u} \cdot \boldsymbol{n} &= \dot{d}, && \text{on} \quad \Sigma \\ \boldsymbol{u} \cdot \boldsymbol{n} &= 0, && \text{on} \quad \Gamma_1 \\ p &= 0, && \text{on} \quad \Gamma_2 \end{aligned}\right. \quad \underset{\text{div}}{\Longrightarrow} \quad \left\{\begin{aligned} -\Delta p &= 0, && \text{in} \quad \Omega^{\mathrm{f}} \\ \frac{\partial p}{\partial \boldsymbol{n}} &= -\rho^{\mathrm{f}} \frac{\partial \boldsymbol{u}}{\partial t} \cdot \boldsymbol{n} = -\rho^{\mathrm{f}} \ddot{d}, && \text{on} \quad \Sigma \\ \frac{\partial p}{\partial \boldsymbol{n}} &= 0, && \text{on} \quad \Gamma_1 \\ p &= 0 && \text{on} \quad \Gamma_2 \end{aligned}\right.$$

- **Physics**: reproduces propagation phenomena
- **Numerics**: explicit coupling unstable

The added-mass operator

Fluid: $$\begin{cases} -\Delta p = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \ddot{d}, & \text{on} \quad \Sigma \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\ p = 0 & \text{on} \quad \Gamma_2 \end{cases}$$

Solid: $$\rho^{\mathrm{s}} \varepsilon \ddot{d} + L d = p_{|\Sigma}, \quad \text{in} \quad \Sigma,$$

Steklov-Poincaré operator

The operator $\mathcal{M}_{\mathrm{A}} : H^{-\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\Sigma)$ defined as: for each $g \in H^{-\frac{1}{2}}(\Sigma)$ we set $\mathcal{M}_{\mathrm{A}}(g) \stackrel{\text{def}}{=} q_{|\Gamma^{\mathrm{w}}}$, where $q \in H^1(\Omega^{\mathrm{f}})$ solves

$$\begin{cases} -\Delta q = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \dfrac{\partial q}{\partial \boldsymbol{n}} = g, & \text{on} \quad \Sigma \\ \dfrac{\partial q}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\ q = 0, & \text{on} \quad \Gamma_2 \end{cases}$$

is a linear, compact, positive and self-adjoint operator in $L^2(\Sigma)$.

From this definition, we have $$p_{|\Sigma} = \mathcal{M}_{\mathrm{A}}(-\rho^{\mathrm{f}} \ddot{d}) = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \ddot{d}$$

The added-mass effect

Fluid:
$$\begin{cases} -\Delta p = 0, & \text{in} \quad \Omega^{\mathrm{f}} \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \ddot{d}, & \text{on} \quad \Sigma \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = 0, & \text{on} \quad \Gamma_1 \\ p = 0 & \text{on} \quad \Gamma_2 \end{cases}$$

Solid:
$$\rho^{\mathrm{s}} \varepsilon \ddot{d} + Ld = p_{|\Sigma}, \quad \text{in} \quad \Sigma, \qquad (1)$$

$$p_{|\Sigma} = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \ddot{d}$$

$$\left(\rho^{\mathrm{s}} \varepsilon + \rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}}\right) \ddot{d} + Ld = 0, \quad \text{in} \quad \Sigma \qquad (2)$$

What kind of time integration scheme of (2) arises from the explicit coupling of (1) ?

Explicit coupling and added-mass

Fluid: $$\begin{cases} \rho^{\mathrm{f}} \dfrac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^{n}}{\delta t} + \nabla p^{n+1} = 0 \\ \operatorname{div} \boldsymbol{u}^{n+1} = 0 \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \dfrac{d^{n} - d^{n-1}}{\delta t} \end{cases} \underset{\text{div}}{\Longrightarrow} \begin{cases} -\Delta p^{n+1} = 0 \\ \dfrac{\partial p^{n+1}}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \dfrac{d^{n} - 2d^{n-1} + d^{n-2}}{\delta t^2} \end{cases}$$

Solid: $$\rho^{\mathrm{s}} \varepsilon \frac{d^{n+1} - 2d^{n} + d^{n-1}}{\delta t^2} + L d^{n} = p^{n+1}_{|\Sigma}$$

$$p^{n+1}_{|\Sigma} = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \frac{d^{n} - 2d^{n-1} + d^{n-2}}{\delta t^2}$$

Condensed FSI problem:

$$\underbrace{\rho^{\mathrm{s}} \varepsilon \frac{d^{n+1} - 2d^{n} + d^{n-1}}{\delta t^2}}_{\textbf{implicit}} + \underbrace{\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \frac{d^{n} - 2d^{n-1} + d^{n-2}}{\delta t^2}}_{\textbf{explicit}} + L d^{n} = 0$$

Explicit coupling yields an explicit discretization of the added mass

An unconditional instability result

Proposition *(Causin-JFG-Nobile 05)*

Let $\lambda_{\max}$ be the largest eigenvalue of $\mathcal{M}_{\mathrm{A}}$ and assume that $L\eta = a\eta$. Then, the previous explicit coupling scheme is unconditionally unstable whenever

$$\frac{\rho^{\mathrm{f}}\lambda_{\max}}{\rho^{\mathrm{s}}\varepsilon} \geq 1. \tag{1}$$

- The instability condition confirms the empirical observations:
  - Instabilities depend on the density ratio
  - The instability condition does not depend on the time step
  - Instabilities occur when the structure is **thin** and **slender (higher $\lambda_{\max}$)**
- Other time schemes have been considered by *Förster-Wall-Ramm 07* with analogous conclusions
- Do not forget that the first assumption to build this toy model was **incompressiblity**

Semi-implicit coupling

Three ideas:

- Treat implicitly the added-mass effect (incompressibility, pressure stress)
- Treat explicitly the fluid domain motion, convective and viscous effects
- Perform this using a projection scheme (Chorin-Teman) within the fluid

(Fernández, JFG, Grandmont, 2007)

The Chorin-Teman projection scheme

- Incompressible Navier-Stokes equations:

$$\rho^{\mathrm{f}}\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}\right)-2\mu\,\mathbf{div}\boldsymbol{\epsilon}(\boldsymbol{u})+\boldsymbol{\nabla}p=\mathbf{0},\quad \text{in}\quad \Omega^{\mathrm{f}}$$
$$\operatorname{div}\boldsymbol{u}=0,\quad \text{in}\quad \Omega^{\mathrm{f}}$$

- **Viscous** step:

$$\begin{cases}\rho^{\mathrm{f}}\left(\dfrac{\widetilde{\boldsymbol{u}}^{n+1}-\boldsymbol{u}^{n}}{\delta t}+\widetilde{\boldsymbol{u}}^{n+1}\cdot\boldsymbol{\nabla}\widetilde{\boldsymbol{u}}^{n+1}\right)-2\mu\operatorname{div}\boldsymbol{\epsilon}(\widetilde{\boldsymbol{u}}^{n+1})=0, & \text{in}\quad \Omega\\ \widetilde{\boldsymbol{u}}^{n+1}=0, & \text{on}\quad \partial\Omega\end{cases}$$

- **Projection** step:

$$\begin{cases}\rho^{\mathrm{f}}\dfrac{\boldsymbol{u}^{n+1}-\widetilde{\boldsymbol{u}}^{n+1}}{\delta t}+\boldsymbol{\nabla}p^{n+1}=0, & \text{in}\quad \Omega\\ \operatorname{div}\boldsymbol{u}^{n+1}=0, & \text{in}\quad \Omega\\ \boldsymbol{u}^{n+1}\cdot\boldsymbol{n}=0, & \text{on}\quad \partial\Omega\end{cases}\quad \underset{\text{div}}{\Longrightarrow}\quad \begin{cases}-\Delta p^{n+1}=-\dfrac{\rho^{\mathrm{f}}}{\delta t}\operatorname{div}\widetilde{\boldsymbol{u}}^{n+1}, & \text{in}\quad \Omega\\ \dfrac{\partial p^{n+1}}{\partial \boldsymbol{n}}=0, & \text{on}\quad \partial\Omega\end{cases}$$

Semi-implicit coupling: explicit part

- Viscous sub-step:

$$\boldsymbol{d}^{\mathrm{f},n+1} = \mathrm{Ext}(\boldsymbol{d}^{n}_{|\widehat{\Sigma}}), \quad \boldsymbol{w}^{n+1} = \frac{\boldsymbol{d}^{\mathrm{f},n+1} - \boldsymbol{d}^{n}}{\delta t}, \quad \Omega^{\mathrm{f},n+1} = (I + \boldsymbol{d}^{\mathrm{f},n+1})(\widehat{\Omega}^{\mathrm{f}}),$$

$$\rho^{\mathrm{f}} \left(\frac{\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{u}^{n}}{\delta t} + (\widetilde{\boldsymbol{u}}^{n+1} - \boldsymbol{w}^{n+1}) \cdot \boldsymbol{\nabla} \widetilde{\boldsymbol{u}}^{n+1} \right) - 2\mu \,\mathbf{div}\, \boldsymbol{\epsilon}(\widetilde{\boldsymbol{u}}^{n+1}) = 0, \quad \text{in} \quad \Omega^{\mathrm{f},n+1}$$

$$\widetilde{\boldsymbol{u}}^{n+1} = \boldsymbol{w}^{n+1}, \quad \text{on} \quad \Sigma^{n+1}$$

  - Fluid domain, viscous and convective effects explicitly treated

Semi-implicit coupling: implicit part

- Fluid projection sub-step (in a known domain):

$$\begin{cases} \rho^{\mathrm{f}} \dfrac{\boldsymbol{u}^{n+1} - \widetilde{\boldsymbol{u}}^{n+1}}{\delta t} + \boldsymbol{\nabla} p^{n+1} = 0, & \text{in} \quad \Omega^{\mathrm{f},n+1} \\ \mathrm{div} \boldsymbol{u}^{n+1} = 0, & \text{in} \quad \Omega^{\mathrm{f},n+1} \\ \boldsymbol{u}^{n+1} \cdot \boldsymbol{n} = \dfrac{\boldsymbol{d}^{n+1} - \boldsymbol{d}^{n}}{\delta t} \cdot \boldsymbol{n}, & \text{on} \quad \Sigma^{n+1} \end{cases} \underset{\mathrm{div}}{\Longrightarrow} \begin{cases} -\Delta p^{n+1} = -\dfrac{\rho^{\mathrm{f}}}{\delta t} \mathrm{div} \widetilde{\boldsymbol{u}}^{n+1}, & \text{in} \quad \Omega^{\mathrm{f},n+1} \\ \dfrac{\partial p^{n+1}}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \dfrac{\boldsymbol{d}^{n+1} - 2\boldsymbol{d}^{n} + \boldsymbol{d}^{n-1}}{\delta t^2}, & \text{on} \quad \Sigma^{n+1} \end{cases}$$

- Solid equation:

$$\begin{cases} \rho^{\mathrm{s}} \dfrac{\boldsymbol{d}^{n+1} - 2\boldsymbol{d}^{n} + \boldsymbol{d}^{n-1}}{\delta t^2} - \mathbf{div} \left(\boldsymbol{F}(\boldsymbol{d}^{n+1}) \boldsymbol{S}(\boldsymbol{d}^{n+1}) \right) = \mathbf{0}, & \text{in} \quad \widehat{\Omega}^{\mathrm{s}} \\ \boldsymbol{F}(\boldsymbol{d}^{n+1}) \boldsymbol{S}(\boldsymbol{d}^{n+1}) \widehat{\boldsymbol{n}} = J(\boldsymbol{d}^{\mathrm{f},n+1}) \boldsymbol{\sigma}(\widetilde{\boldsymbol{u}}^{n+1}, p^{n+1}) \boldsymbol{F}(\boldsymbol{d}^{\mathrm{f},n+1})^{-\mathrm{T}} \widehat{\boldsymbol{n}}, & \text{on} \quad \widehat{\Sigma} \end{cases}$$

- ▸ Projection sub-step in a fixed fluid domain
- ▸ Implicit part solved with much cheaper inner iterations

A stability result (linear case)

Proposition: *(Fernandez-JFG-Grandmont 2007)*

Assume the interface matching operator to be L^2-stable. Then, under condition

$$\rho^{\mathrm{s}} \geq C\left(\rho^{\mathrm{f}}\frac{h}{H^\alpha} + 2\frac{\mu\delta t}{hH^\alpha}\right), \quad \text{with} \quad \alpha \overset{\text{def}}{=} \begin{cases} 0, & \text{if} \quad \overline{\Omega^{\mathrm{s}}} = \Sigma, \\ 1, & \text{if} \quad \overline{\Omega^{\mathrm{s}}} \neq \Sigma, \end{cases}$$

the following discrete energy inequality holds:

$$\frac{1}{\delta t}\left[\frac{\rho^{\mathrm{f}}}{2}\|\boldsymbol{u}_h^{n+1}\|^2_{0,\Omega^{\mathrm{f}}} - \frac{\rho^{\mathrm{f}}}{2}\|\boldsymbol{u}_h^{n}\|^2_{0,\Omega^{\mathrm{f}}} + \frac{\rho^{\mathrm{s}}}{2}\left\|\frac{\boldsymbol{d}_H^{n+1} - \boldsymbol{d}_H^{n}}{\delta t}\right\|^2_{0,\Omega^{\mathrm{f}}} - \frac{\rho^{\mathrm{s}}}{2}\left\|\frac{\boldsymbol{d}_H^{n} - \boldsymbol{d}_H^{n-1}}{\delta t}\right\|^2_{0,\Omega^{\mathrm{f}}}\right]$$
$$+ \frac{1}{2\delta t}\left[a^{\mathrm{s}}(\boldsymbol{d}_H^{n+1}, \boldsymbol{d}_H^{n+1}) - a^{\mathrm{s}}(\boldsymbol{d}_H^{n}, \boldsymbol{d}_H^{n})\right] + \mu\|\epsilon(\widetilde{\boldsymbol{u}}_h^{n+1})\|^2_{0,\Omega^{\mathrm{f}}} \leq 0$$

Therefore, the semi-implicit coupling scheme is conditionnally stable in the energy norm.

Navier-Sokes / nonlinear shell coupling

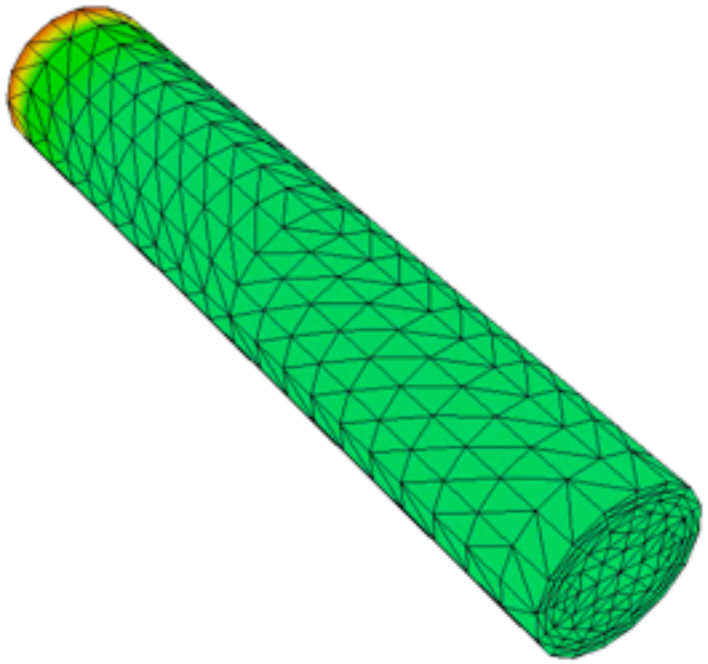
- Straight cylinder: 50 time steps of length $\delta t = 2 \times 10^{-4} s$

COUPLING	ALGORITHM	CPU time
Implicit	FP-Aitken	24.86
	quasi-Newton	6.05
	Newton	4.77
Semi-Implicit	Newton	1

2001 (FP-Aitken)

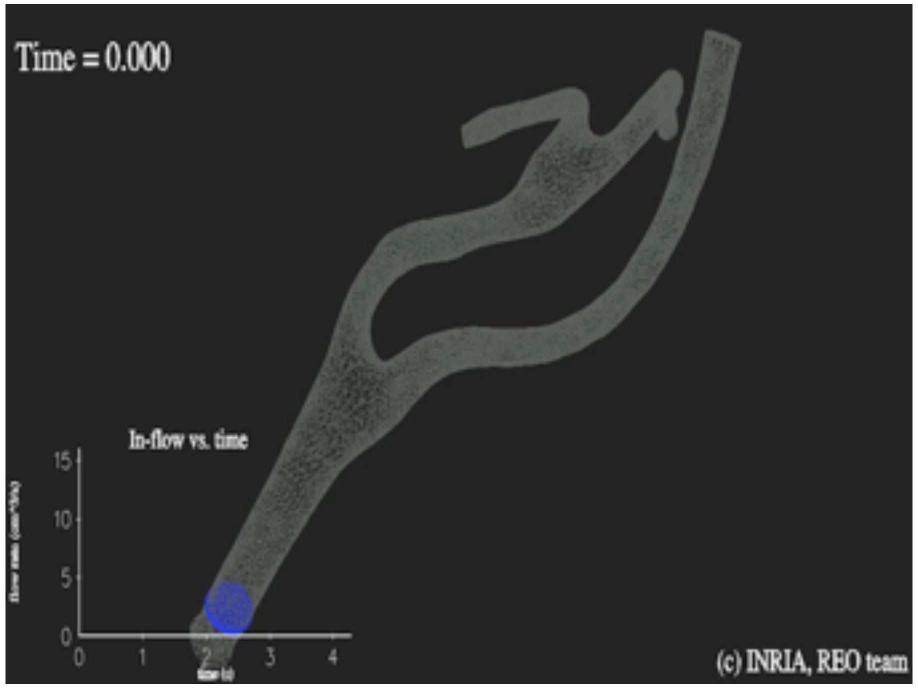
2003 (quasi-Newton, Newton)

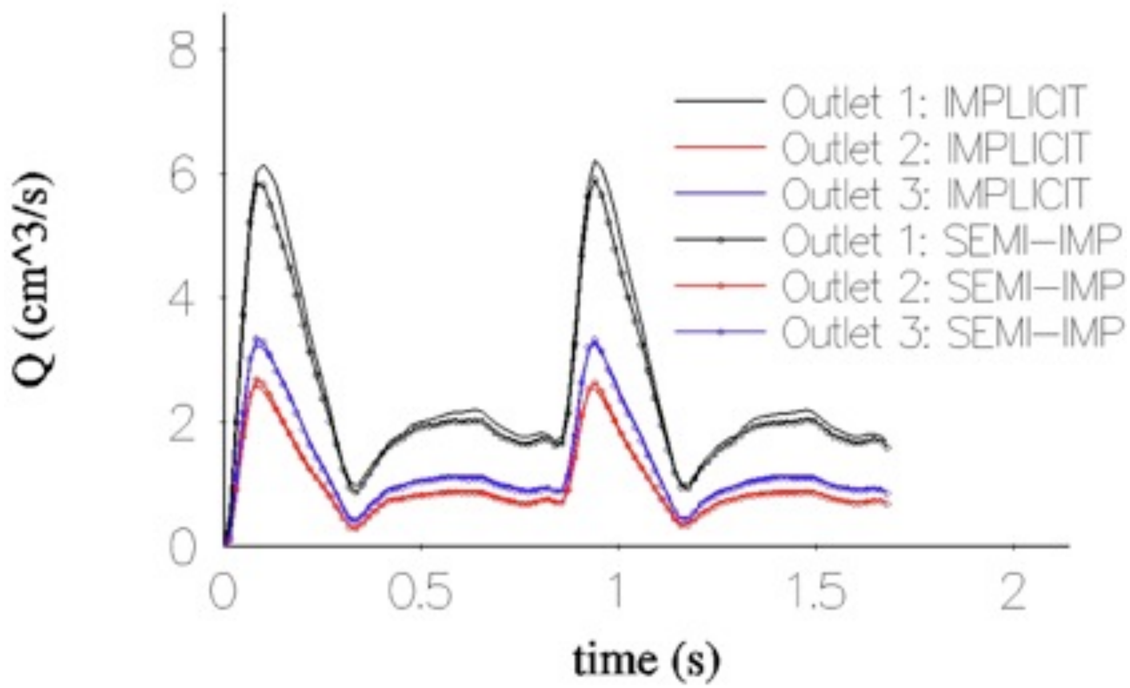
2007 (Semi-Implicit Newton)



Navier-Sokes / Nonlinear shell coupling

- Carotid artery (in-vivo model): 9 cardiac cycles, 4500 times steps
  - $\delta t = 1.68 \times 10^{-3} s$
  - Fluid: 70047 Tetrahedra ($\mathbb{P}_1/\mathbb{P}_1$ FE)
  - Solid: 8103 Quadrilaterals (MITC4 FE)
  - Parameters: $\mu = 0.035\, poise$, $\rho^f = 1\, g/cm^3$, $\rho^s = 1.2\, g/cm^3$, $E = 6 \times 10^6\, dynes/cm^2$, $\nu = 0.3$.





Out-flow

COUPLING	CPU time
Implicit	**6.7**
Semi-Implicit	**1.0**

Dimensionless CPU time

Recent approaches: explicit schemes

- Idea: only solid inertia needs to be implicitly coupled to the fluid
- Fluid

$$\begin{cases} \rho^{\mathrm{f}} \partial_t \boldsymbol{u} - \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u}, p) = \mathbf{0} & \text{in} \quad \Omega^{\mathrm{f}} \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega^{\mathrm{f}} \\ \boldsymbol{u} = \dot{\boldsymbol{d}} & \text{on} \quad \Sigma \end{cases}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} + \rho^{\mathrm{s}} \epsilon \partial_t \boldsymbol{u} = -\boldsymbol{L}^{\mathrm{e}} \boldsymbol{d} \quad \text{on} \quad \Sigma$$

- Thin solid

$$\begin{cases} \rho^{\mathrm{s}} \epsilon \partial_t \dot{\boldsymbol{d}} + \boldsymbol{L}^{\mathrm{e}} \boldsymbol{d} = -\boldsymbol{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} & \text{on} \quad \Sigma \\ \dot{\boldsymbol{d}} = \partial_t \boldsymbol{d} & \text{on} \quad \Sigma \end{cases}$$

$$\boxed{\boldsymbol{\sigma}(\boldsymbol{u}^n, p^n)\boldsymbol{n} + \frac{\rho^{\mathrm{s}} \epsilon}{\tau} \boldsymbol{u}^n = \frac{\rho^{\mathrm{s}} \epsilon}{\tau} \dot{\boldsymbol{d}}^{n-1} - \boldsymbol{L}^{\mathrm{e}} \boldsymbol{d}^{\star}} \quad \text{on} \quad \Sigma, \qquad \boldsymbol{d}^{\star} = \begin{cases} 0 \\ \boldsymbol{d}^{n-1} \\ \boldsymbol{d}^{n-1} + \tau \dot{\boldsymbol{d}}^{n-1} \end{cases}$$

- Added-mass free *and* parameter free
- **Key issue is now the accuracy !**

Glowinski, Canic, et al. 2009
Fernández 2012

Outline

- Forward problem in Fluid-Structure Interaction
- Inverse problem in Fluid-Structure Interaction

Medical data assimilation

Models

- Navier-Stokes equations
- Solid mechanics
- ...

Measurements

- medical imaging (CT, MRI, ...)
- blood flow (US, PC-MRI,...)
- pressure (catheter)
- ...

Estimate parameters

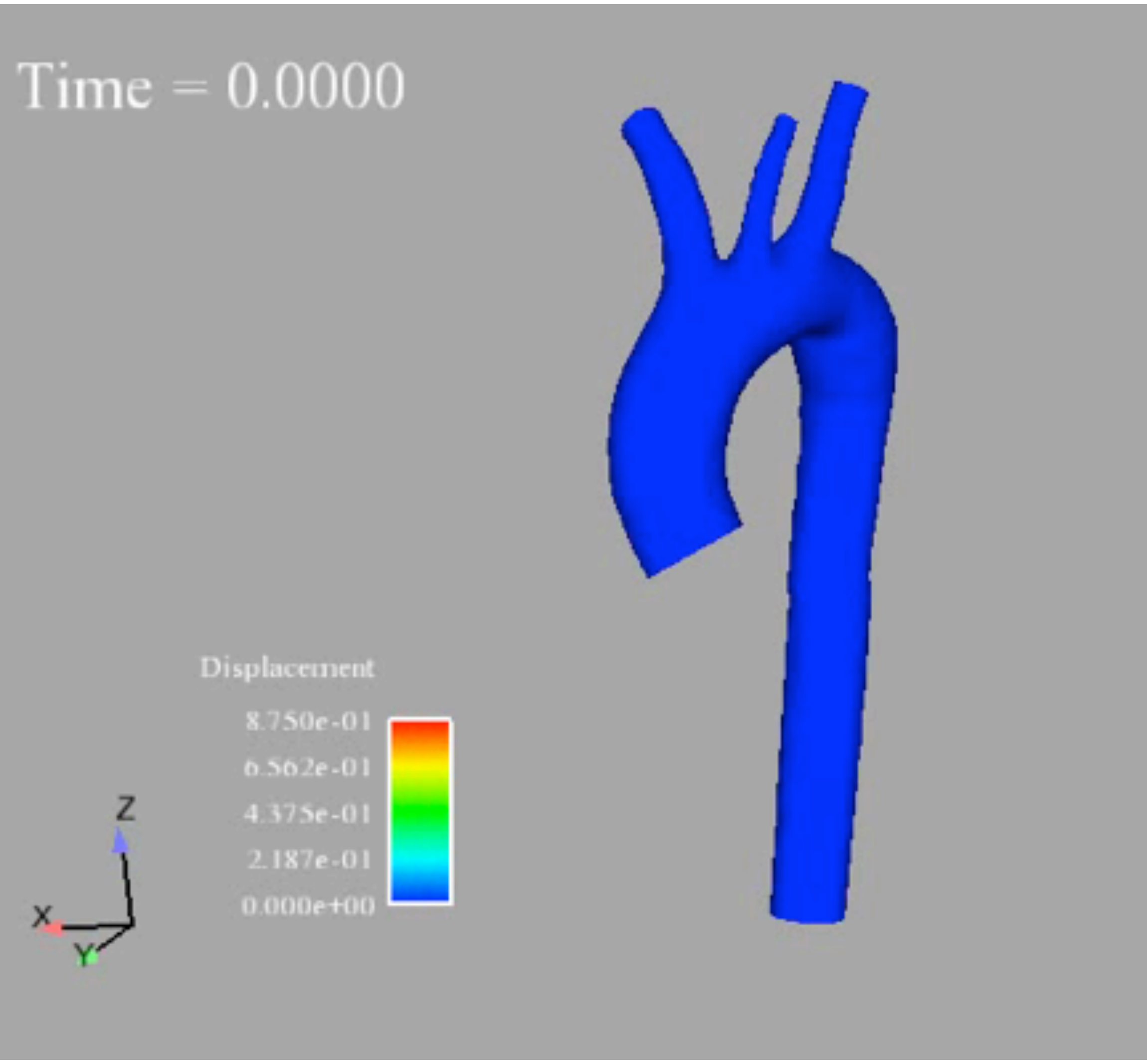
- artery wall stiffness
- boundary condition
- ...

Access to hidden quantities

- pressure
- wall stress

Improve measurements

- regularization
- interpolation
- ...



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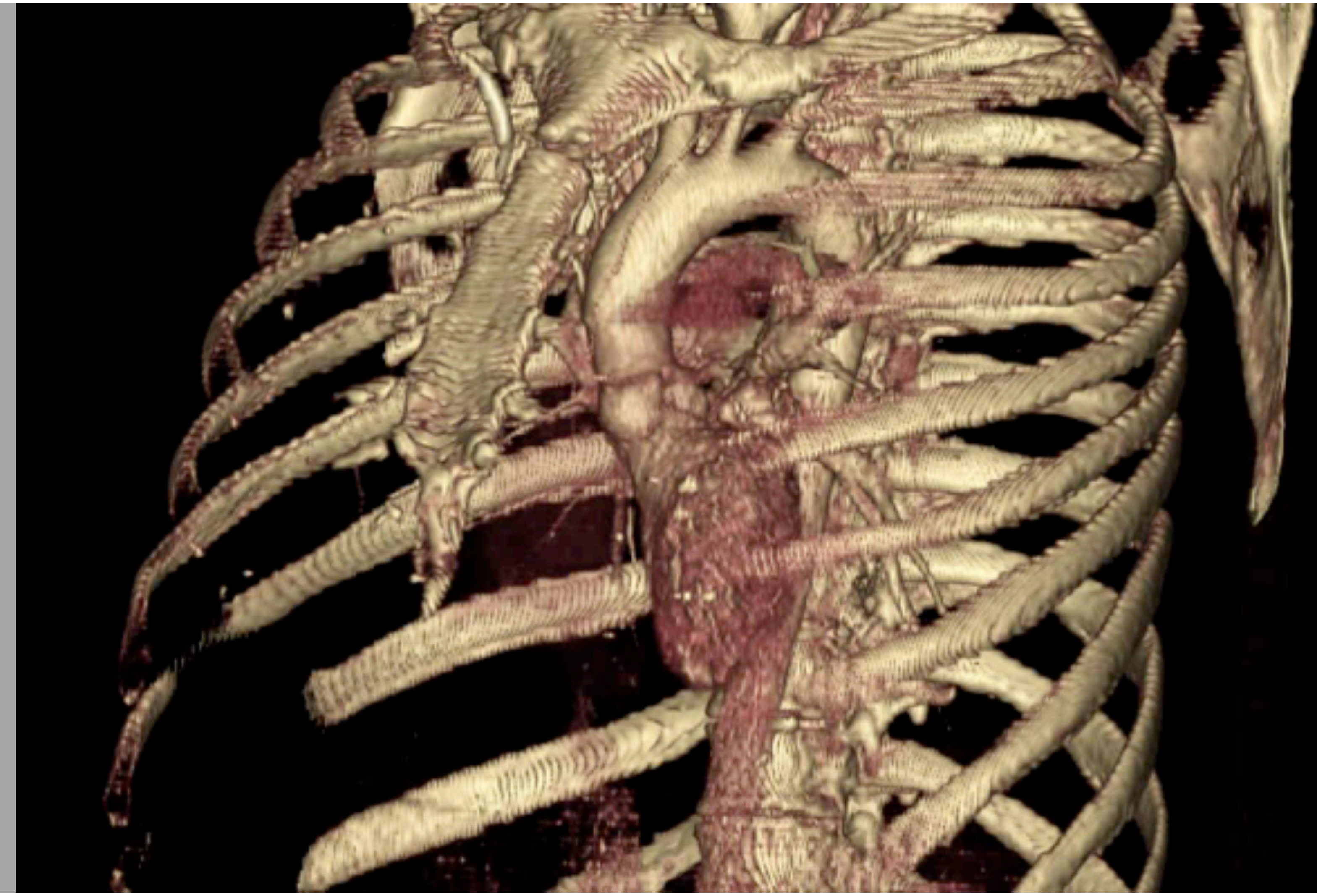


CVBRL, Stanford

Bertoglio, Chapelle, Fernandez, JFG, Moireau, 2013

Moireau, Bertoglio, Xiao, Figueroa, Taylor, Chapelle, JFG, 2012

Bertoglio, Moireau, JFG, 2013

Introduction to Luenberger observers

- Dynamical system: $\begin{cases} \dfrac{dX}{dt} = A(X, \theta) \\ X(0) = X_0 \end{cases}$

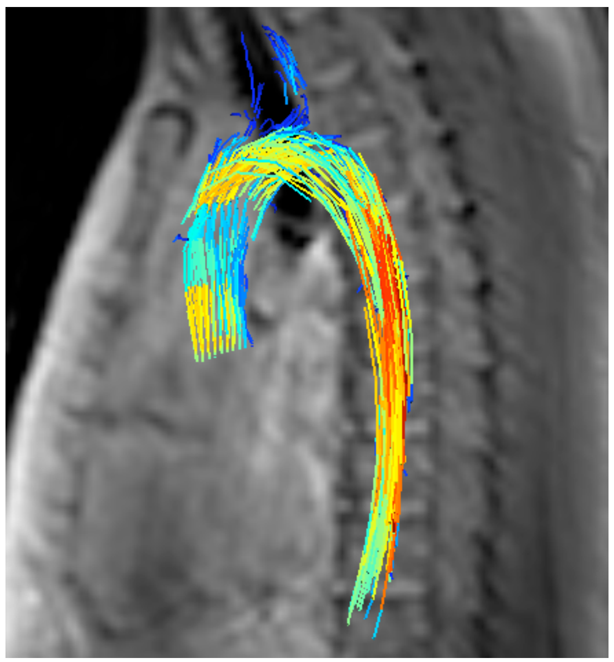
- Example of **state variable**: X=$[\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{v}]$

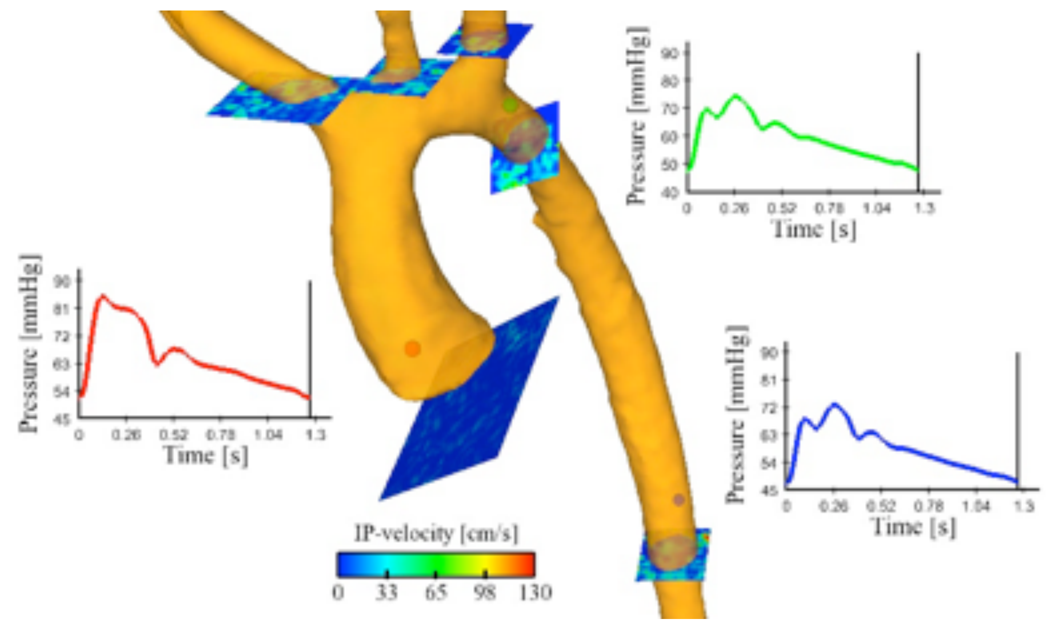
- Example of **parameters**: θ = [Young modulus, boundary conditions, $\ldots$]

Imperfect knowledge of $X(t = 0)$ and θ: $\hat{X}_0$ and $\hat{\theta}_0$

Introduction to Luenberger observers

- Partial **observations** of X: $\quad Z = H(X)$





Data: I. Valverde, P. Beerbaum (euHeart project).

State estimation

Parameters identification

- Minimize

$$J(X_0,\theta) = \frac{1}{2}\int_0^T \|Z - H(X(t))\|_W^2\,dt + \frac{1}{2}\|X_0 - \hat{X}_0\|_P^2 + \frac{1}{2}\|\theta - \hat{\theta}_0\|_P^2$$

where $X(t)$ is the solution of the state equation associated to (X_0,θ).

Data assimilation

- **Variational approach:**
  - Optimization algorithms
  - Usually based on gradient (adjoint equations)

In **hemodynamics**:

Piccinelli, Mirabella, Passerini, Haber, Veneziani, 2012
D'Elia, Perego, Veneziani, 2012
Perego, Veneziani, Vergara, 2012

- **Filtering approach:**
  - Sequential correction of the state and the parameters

In **hemodynamics**:

Moireau, Bertoglio, Xiao, Figueroa, Taylor, Chapelle, JFG, 2012
Bertoglio, Chapelle, Fernandez, JFG, Moireau, 2013
Bertoglio, Moireau, JFG, 2013

Strategy : reduced filtering

- Kalman filtering (UKF) is only used for the **parameters** θ ($p \ll N$)
- A much cheaper filter (Luenberger) is used for the **state** X

Introduction to Luenberger observers

- In this talk: **only state estimation**

$$J(X_0) = \frac{1}{2}\int_0^T \|Z - H(X(t))\|_W^2 \, dt + \frac{1}{2}\|X_0 - \hat{X}_0\|_P^2$$

- In dissipative system, error in initial condition is “forgotten”....

- ... but, in view of **joint state-parameter** estimation,
  we want to forget it **as quickly as possible** !

Introduction to Luenberger observers

- **Sequential estimation**
  - introduce a modified system: the "**observer**"

$$\left\{ \begin{array}{rcl} \dfrac{d\hat{X}}{dt} & = & A(\hat{X}) + G(Z - H(\hat{X})) \\ \hat{X}(0) & = & \hat{X}_0 \end{array} \right.$$

  - with the ultimate objective to converge to the real trajectory *X(t)*

Introduction to Luenberger observers

- Search for the filter G such that the **optimality criterion** is satisfied:

$$X(t) = X_{[\mathrm{argmin}_{J(\cdot,t)}]}$$

  G obtained from the **Riccati** or **HJB** equations.

  ➡ Intractable for PDEs

- Cheaper alternative:
  - renounce to the optimality criterion
  - build an *ad hoc* operator G to have the error decreased
- Idea introduced by Luenberger in 1963.
- Also known as "nudging" in the data assimilation community

Hoke-Anthes 1976, Stauffer-Seaman 1990, Auroux-Blum 2005,...

Introduction to Luenberger observers

Luenberger filter: looks simple but...

- Sometimes, there are pitfalls
- There is room for creativity!

The case of a linear dynamics:

- "Real" dynamics (without noise): $\dfrac{dX}{dt} = AX + G\underbrace{(Z - HX)}_{=0}$
- Observer *(Luenberger)*: $\dfrac{d\hat{X}}{dt} = A\hat{X} + G(Z - H\hat{X})$
- Dynamics of the error $e_X = X - \hat{X}$:

$$\frac{\mathrm{d}e_X}{\mathrm{d}t} = (A - GH)e_X$$

Introduction to Luenberger observers

- Spectral properties of the error dynamics:

$$(A - GH)\Phi_k = \lambda_k \Phi_k$$

≤ 0

Goal: Devise an operator G to reduce $\max\left(Re(\lambda_k)\right)$

- Typically, to decrease the initial error by a factor β in a time T_c:

$$\max\left(Re(\lambda_k)\right) \leq \frac{\log \beta}{T_c}$$

- Ex: to have $\beta = 10$ in $T_c = 0.1s$, $\max\left(Re(\lambda_k)\right) \approx -25$

Luenberger observers in elastodynamics

- Elastodynamics equations $X = [\boldsymbol{d}, \boldsymbol{v}]$

- Velocity filtering: *Direct Velocity Feedback* (**DVF**)
(Moireau-Chapelle-Le Tallec, 2008)

$$\int_{\Sigma_0} (\boldsymbol{z} - \hat{\boldsymbol{v}}) \cdot \boldsymbol{\phi}_i$$

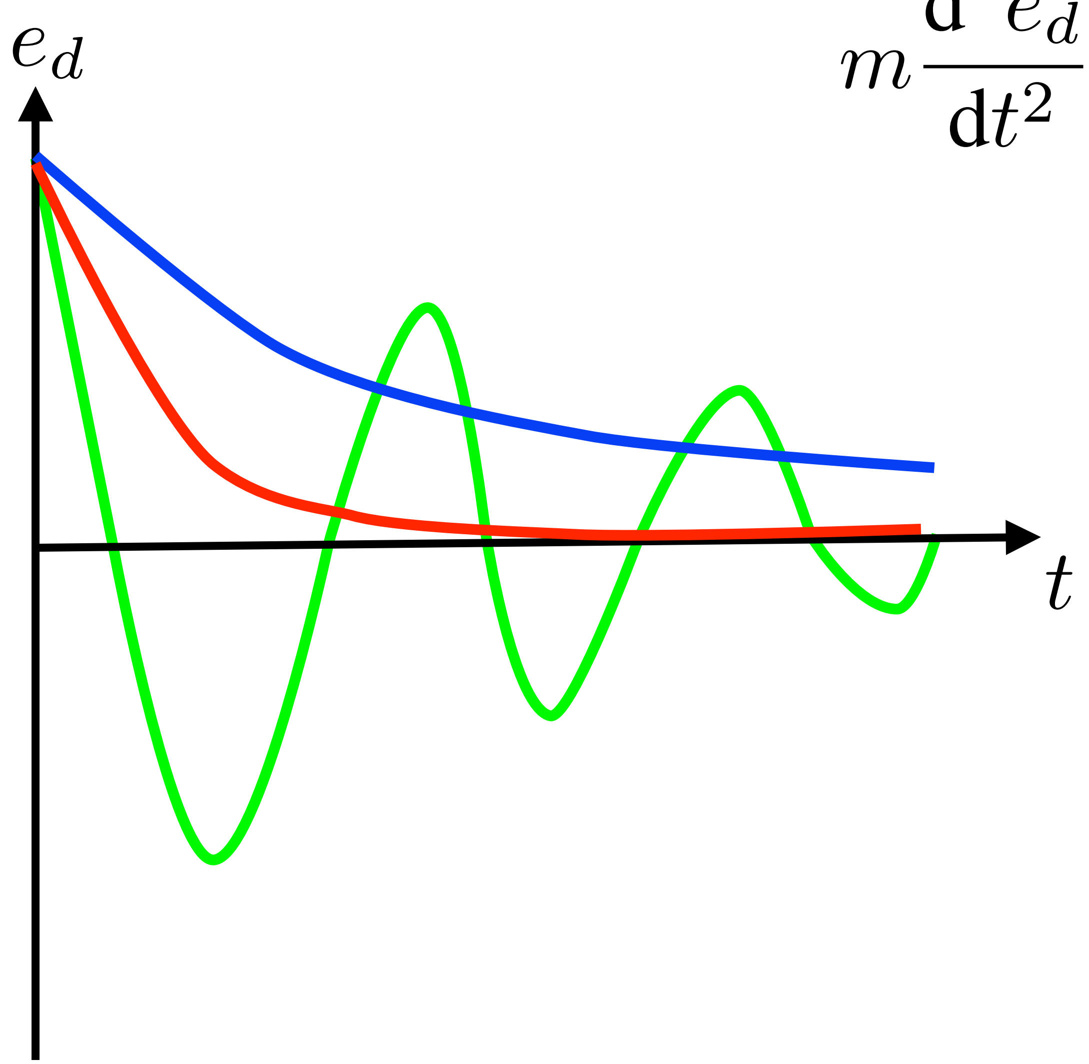
$$\begin{cases} M_s \dfrac{\mathrm{d}\hat{\boldsymbol{v}}}{\mathrm{d}t} + K_s \hat{\boldsymbol{d}} & = & R + \gamma_v \overbrace{H^T M_H (Z - H\hat{\boldsymbol{v}})} \\ \dfrac{\mathrm{d}\hat{\boldsymbol{d}}}{\mathrm{d}t} & = & \hat{\boldsymbol{v}} \end{cases}$$

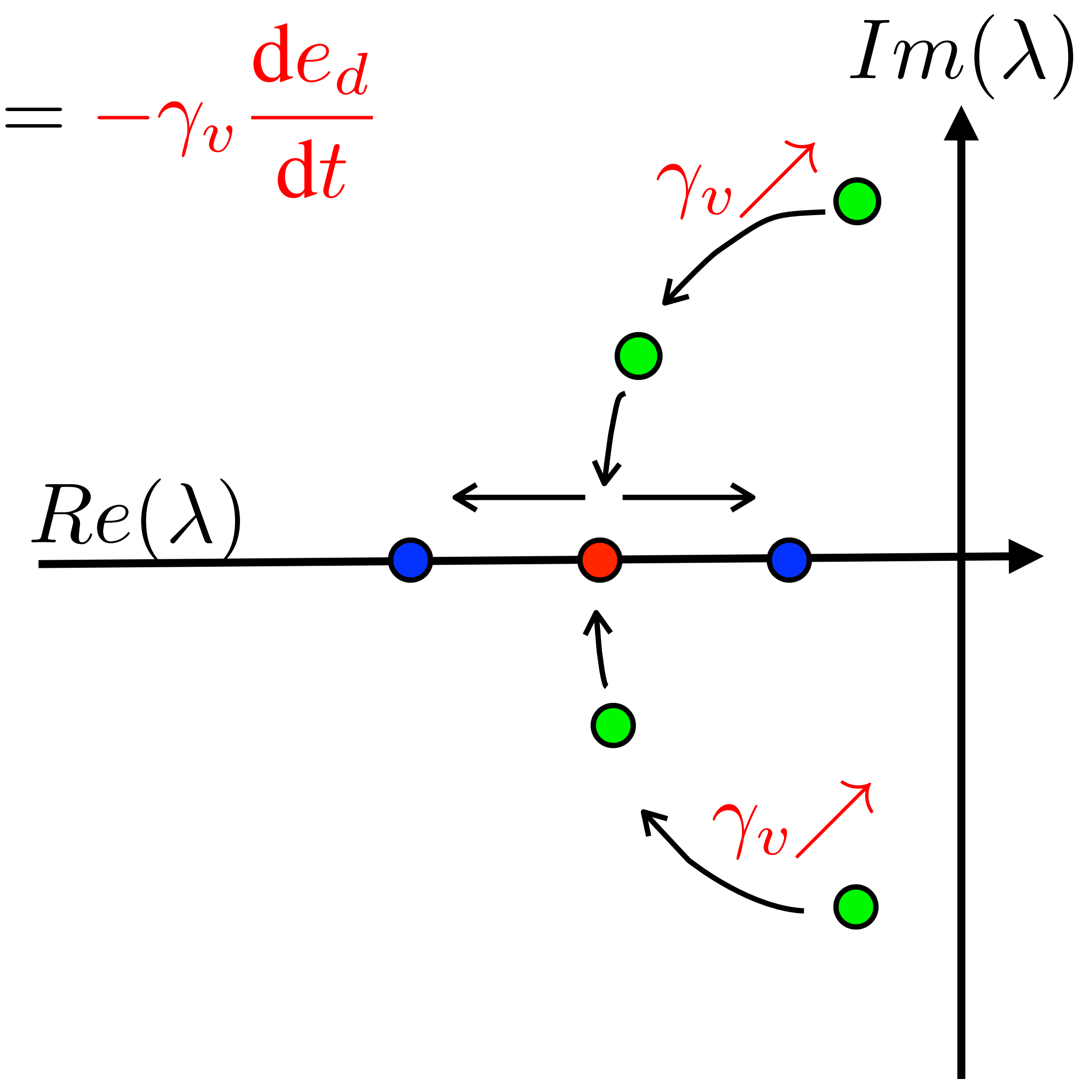
- Equation of the error: $e_{\boldsymbol{v}} = \boldsymbol{v} - \hat{\boldsymbol{v}}$, $e_{\boldsymbol{d}} = \boldsymbol{d} - \hat{\boldsymbol{d}}$

$$M_s \frac{\mathrm{d}e_{\boldsymbol{v}}}{\mathrm{d}t} + K_s e_{\boldsymbol{d}} = -\gamma_v H^T M_H H e_{\boldsymbol{v}}$$

- A trivial example: linear oscillator

$$m\frac{\mathrm{d}^2 e_d}{\mathrm{d}t^2} + k e_d = -\gamma_v \frac{\mathrm{d}e_d}{\mathrm{d}t}$$





- Let $\omega_0 = \sqrt{\frac{k}{m}}$ and $\beta = \frac{\gamma_v}{2m}$

  – If $\beta < \omega_0$: underdamped

  – If $\beta = \omega_0$: critically damped

  – If $\beta > \omega_0$: overdamped

Luenberger observers in elastodynamics

- If the measurements are **displacements**

- First option:

$$\left\{\begin{array}{rcl} M_s \dfrac{\mathrm{d}\hat{\boldsymbol{v}}}{\mathrm{d}t} + K_s \hat{\boldsymbol{d}} & = & R + \gamma_d H^T M_H (Z - H\hat{\boldsymbol{d}}) \\ \dfrac{\mathrm{d}\hat{\boldsymbol{d}}}{\mathrm{d}t} & = & \hat{\boldsymbol{v}} \end{array}\right.$$

- **Remarks:**
  - Related to the "**Image Force Method**" used in the medical imaging community
  - Poor behavior (except for systems with very large dissipation)

Luenberger observers in elastodynamics

- Displacement filtering: *Schur Displacement Feedback* **(SDF)**
  (Moireau-Chapelle-Le Tallec, 2009)

$$\begin{cases} M_s \dfrac{\mathrm{d}\hat{\boldsymbol{v}}}{\mathrm{d}t} + K_s \hat{\boldsymbol{d}} & = & R \\ \dfrac{\mathrm{d}\hat{\boldsymbol{d}}}{\mathrm{d}t} & = & \hat{\boldsymbol{v}} + \gamma_d K_\mu^{-1} H^T M_H (Z - H(\hat{\boldsymbol{d}})) \end{cases}$$

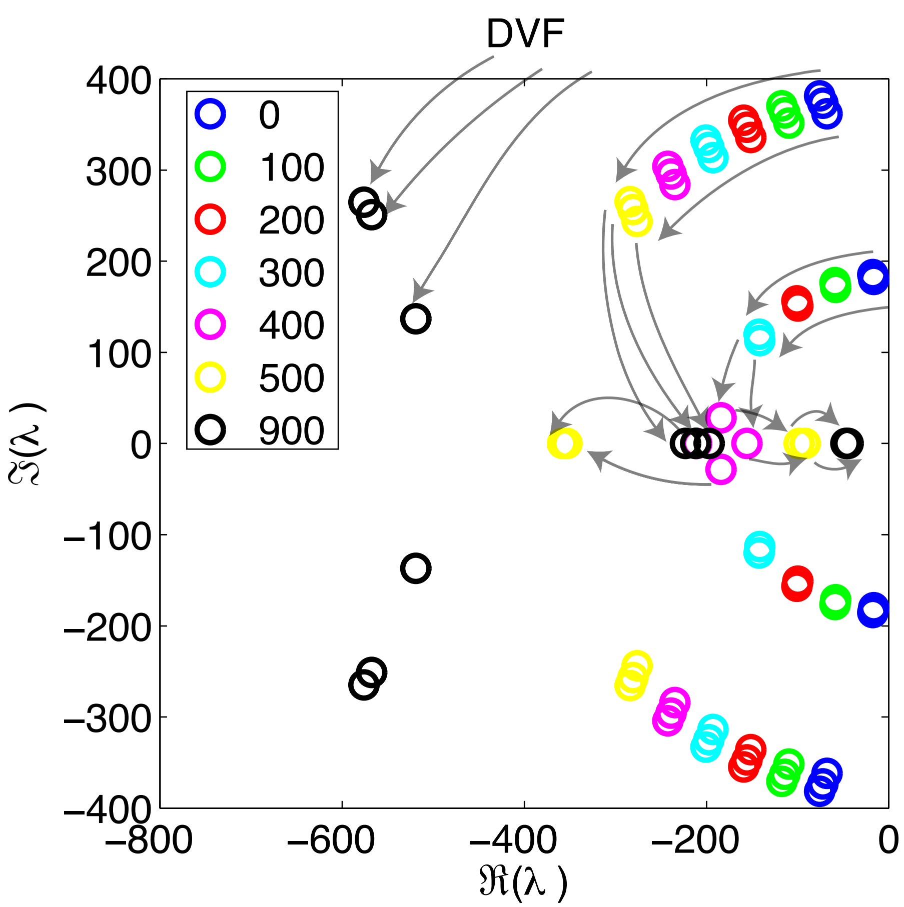
with $K_\mu = K_s + \mu H^T M_\Gamma H$.

- **Remarks:**
  - Velocity is no longer the derivative of displacement

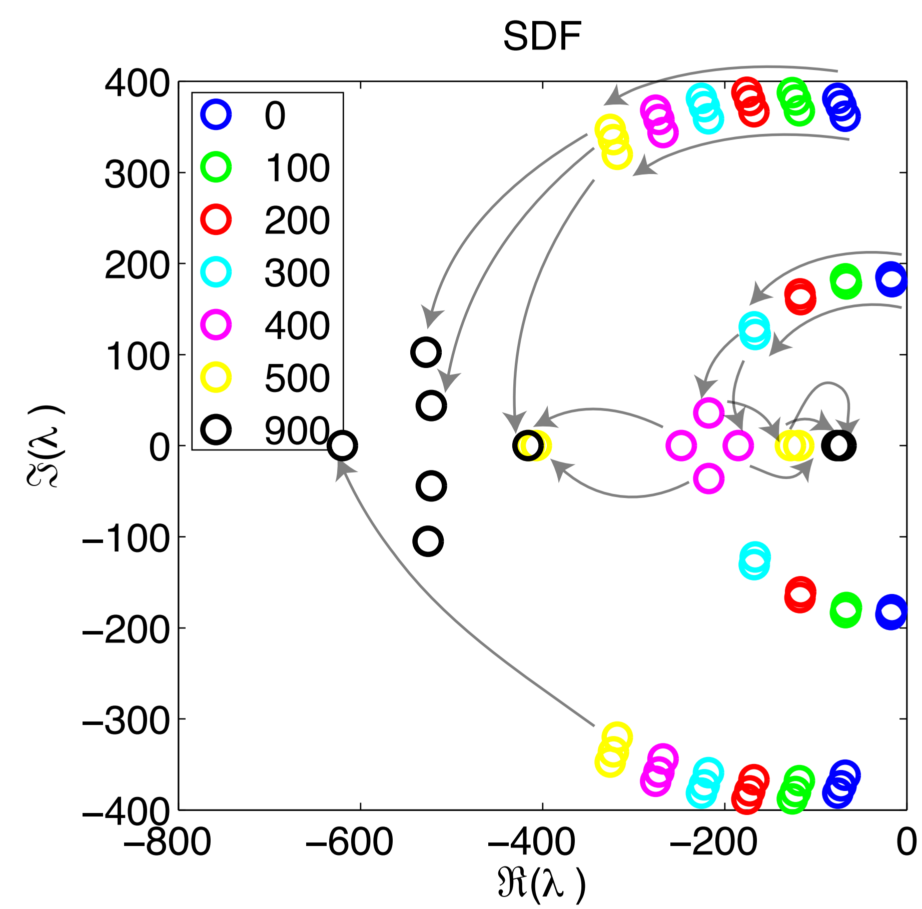
$$\frac{\partial \boldsymbol{d}}{\partial t} = \boldsymbol{v} + \gamma_d \mathrm{Ext}(\boldsymbol{z} - \boldsymbol{d})$$

  - **The norm matters!**

Luenberger observers in elastodynamics



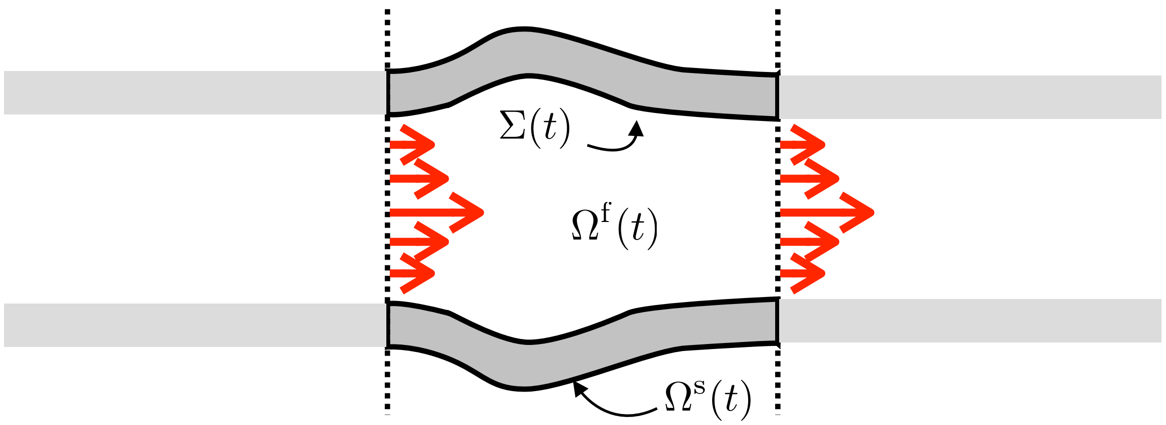
Velocity feedback



Displacement feedback

➡ **DVF and SDF have a similar behavior in elastodynamics**

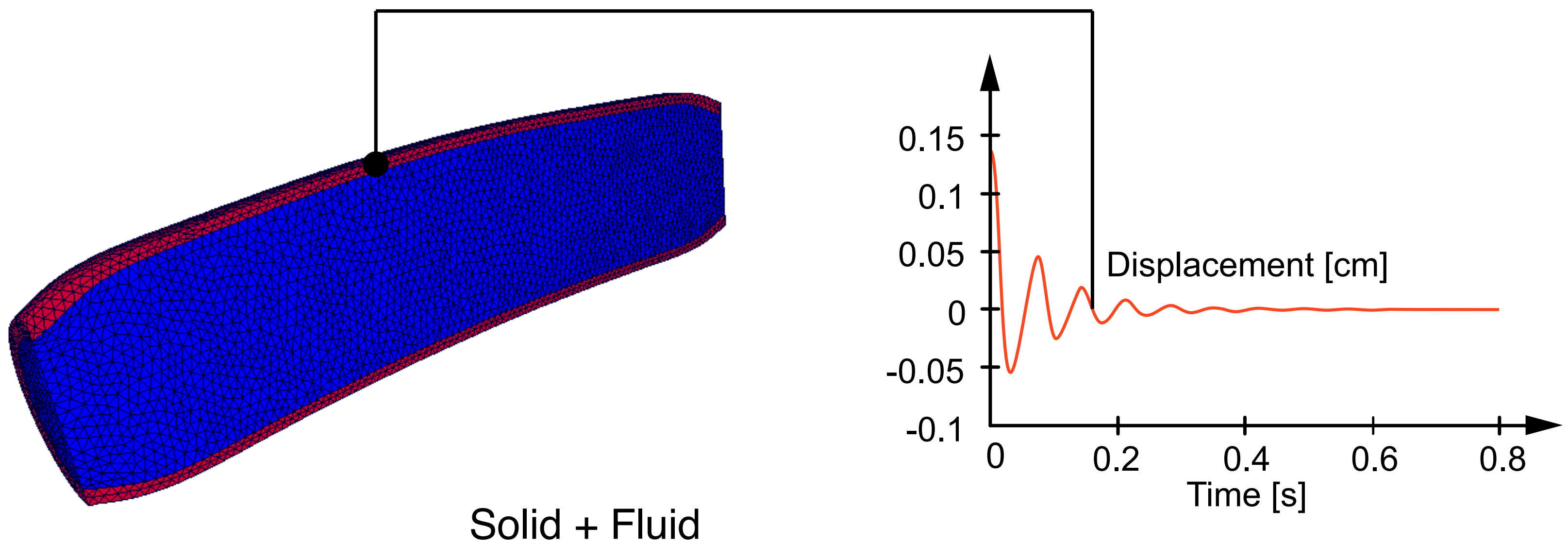
Luenberger observers in FSI

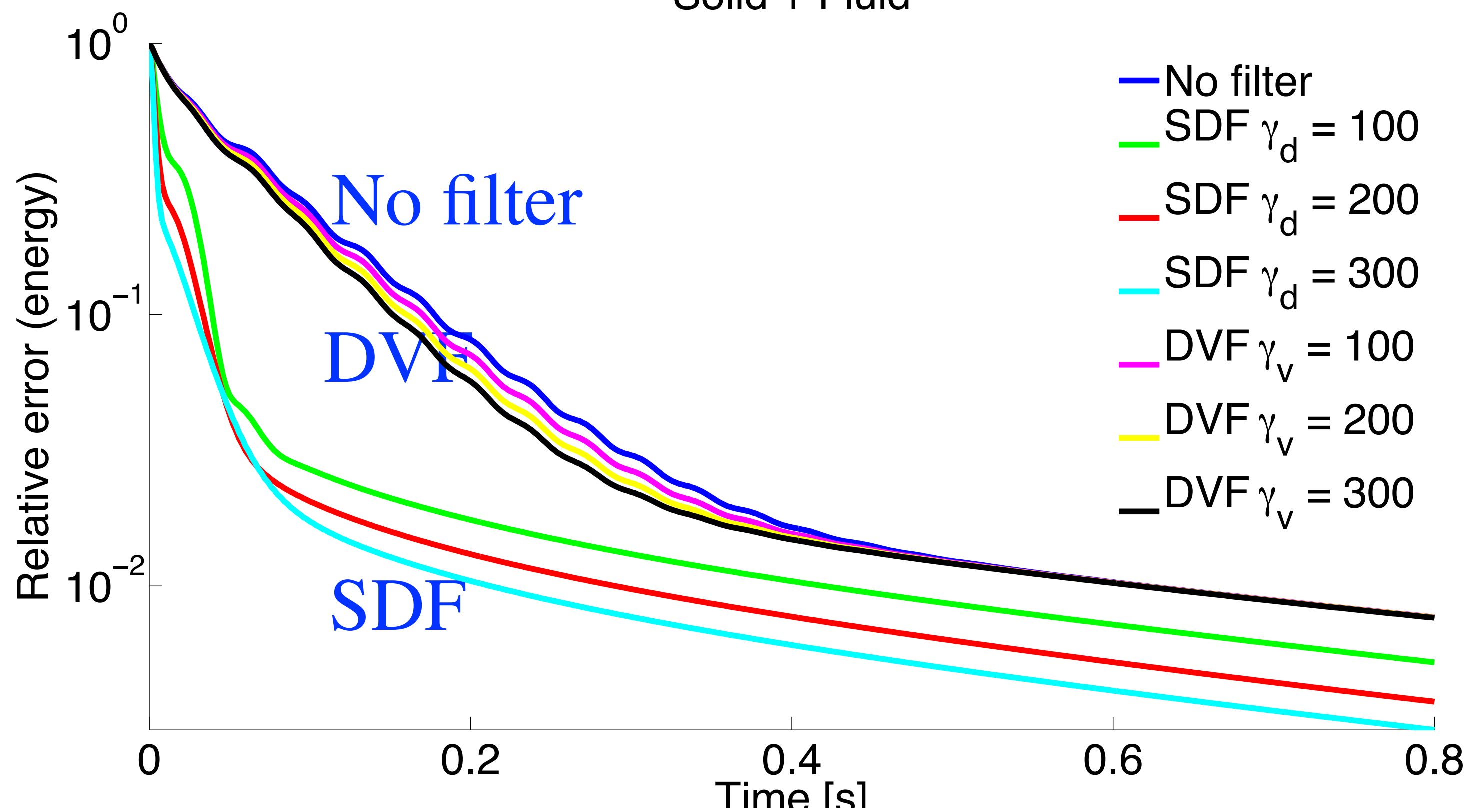


- We limit ourselves to ***solid measurements***
- We are interested in:
  - The effect of the FSI coupling
  - The effect of boundary conditions
  - The effect of fluid dissipation

1st nonlinear test: stabilization to equilibrium

- Fluid initially at rest
- Initial perturbation in the solid

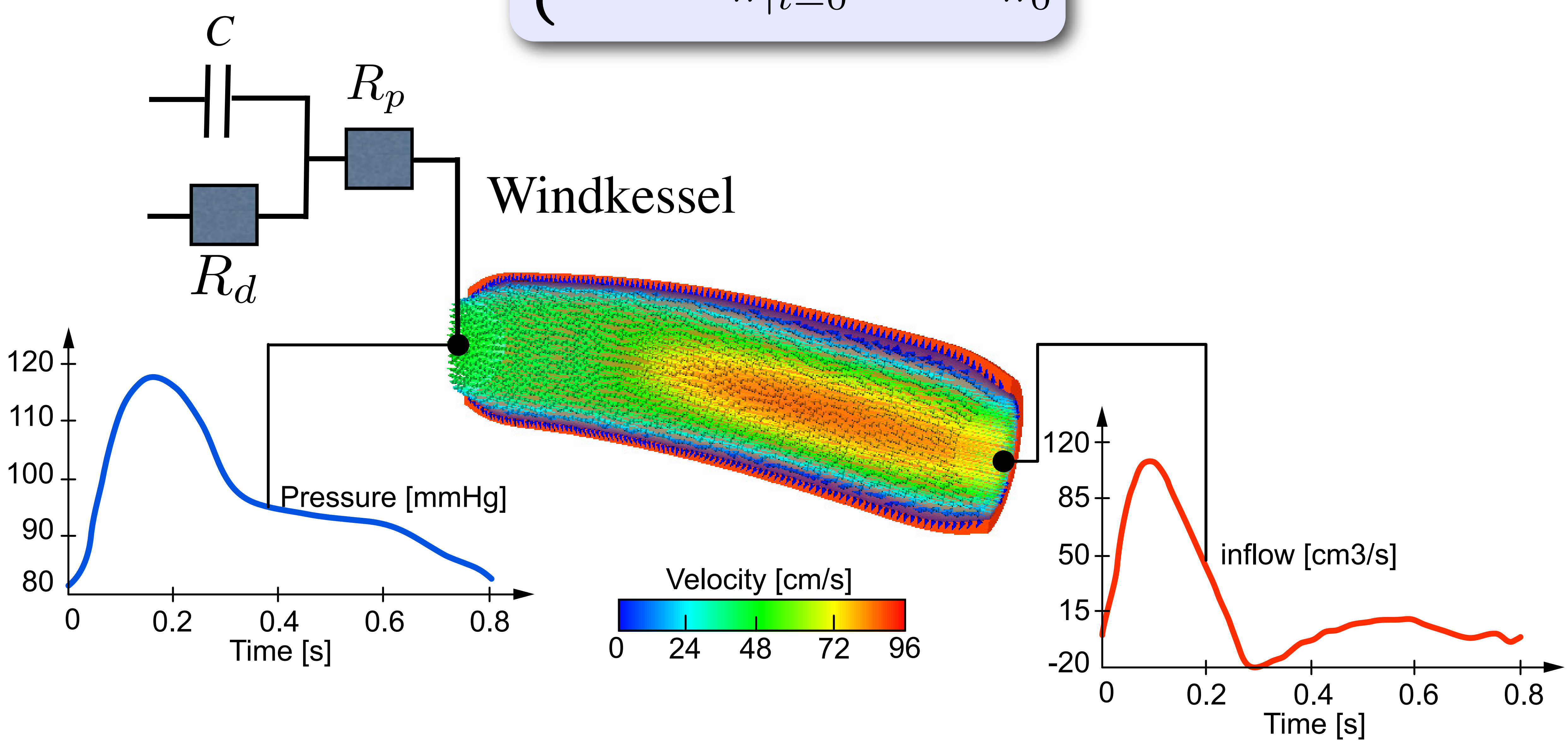


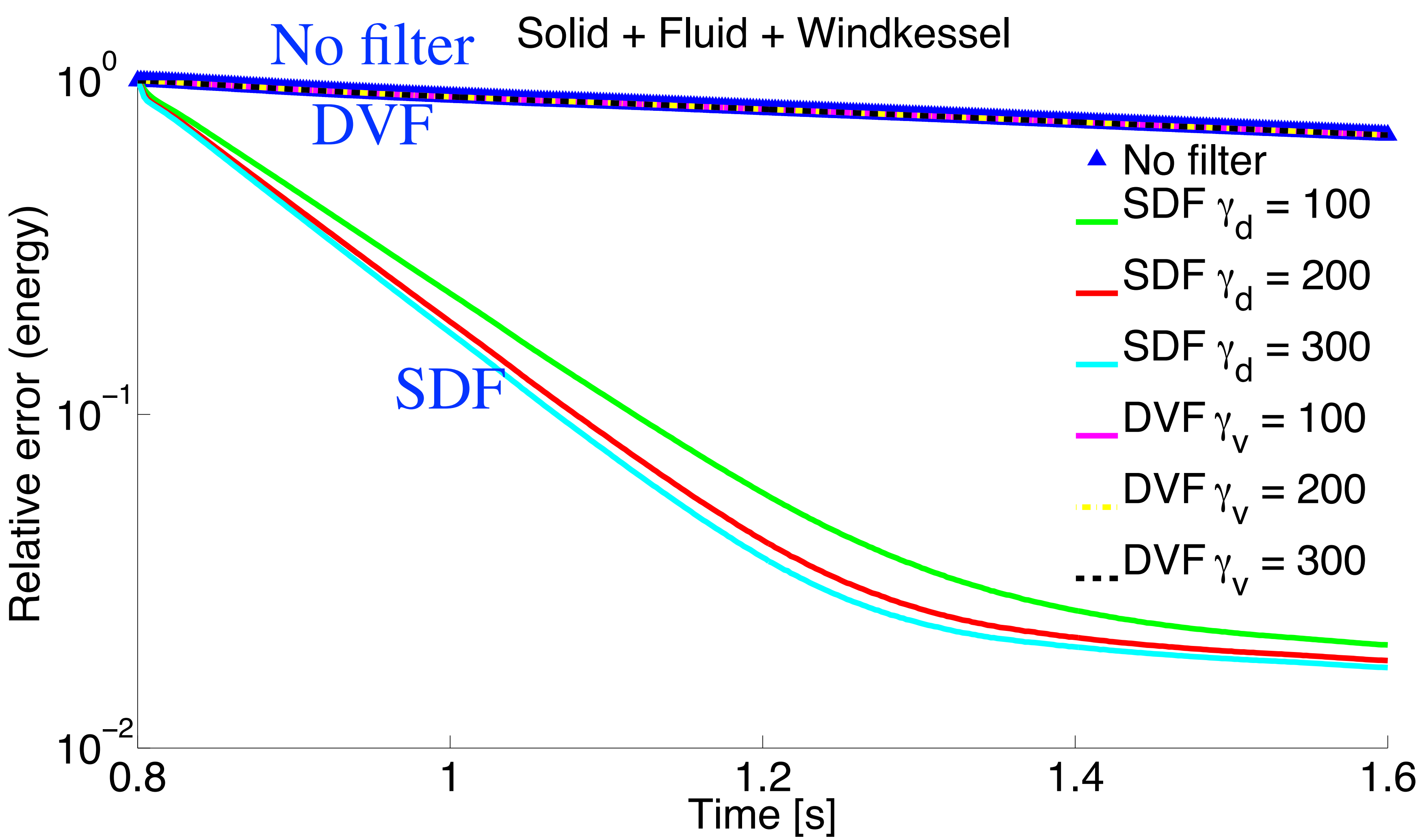


Energy norm of the error

2d nonlinear test: hemodynamics

$p = \pi + R_p Q$ with $\begin{cases} C\dfrac{d\pi}{dt} + \dfrac{\pi}{R_d} = Q \\ \pi|_{t=0} = \pi_0 \end{cases}$ and $Q = \int_{\Gamma_{\text{out}}} \boldsymbol{u} \cdot \boldsymbol{n} \, dS$





Energy norm of the error

SDF and DVF in FSI

Analysis of a toy model

- Simplified fluid:

$$\begin{cases} \rho^{\mathrm{f}} \dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} p = 0, \text{ in } \Omega^{\mathrm{f}} \\ \operatorname{div} \boldsymbol{u} = 0, \text{ in } \Omega^{\mathrm{f}} \\ \boldsymbol{u} \cdot \boldsymbol{n} = \dot{\boldsymbol{d}}, \text{ on } \Sigma \end{cases} \xrightarrow[\text{div}]{} \begin{cases} -\Delta p = 0, \text{ in } \Omega^{\mathrm{f}} \\ \dfrac{\partial p}{\partial \boldsymbol{n}} = -\rho^{\mathrm{f}} \dfrac{\partial \boldsymbol{u}}{\partial t} \cdot \boldsymbol{n} = -\rho^{\mathrm{f}} \ddot{\boldsymbol{d}} \cdot \boldsymbol{n}, \text{ on } \Sigma \end{cases}$$

- Let $\mathcal{M}_{\mathrm{A}}$ be the "Neumann-to-Dirichlet" operator: $p_{|\Sigma} = -\rho^{\mathrm{f}} \mathcal{M}_{\mathrm{A}} \ddot{\boldsymbol{d}} \cdot \boldsymbol{n}$

- Linear elasticity:

$$\begin{cases} \rho^{\mathrm{s}} \ddot{\boldsymbol{d}} - \operatorname{div} \sigma(\boldsymbol{d}) = 0, \text{ in } \Omega^{\mathrm{s}} \\ \sigma(\boldsymbol{d}) \cdot \boldsymbol{n} = p|_{\Sigma} \boldsymbol{n} = -\rho^{f} \mathcal{M}_{A} \ddot{\boldsymbol{d}} \cdot \boldsymbol{n} \, \boldsymbol{n}, \text{ on } \Sigma \end{cases}$$

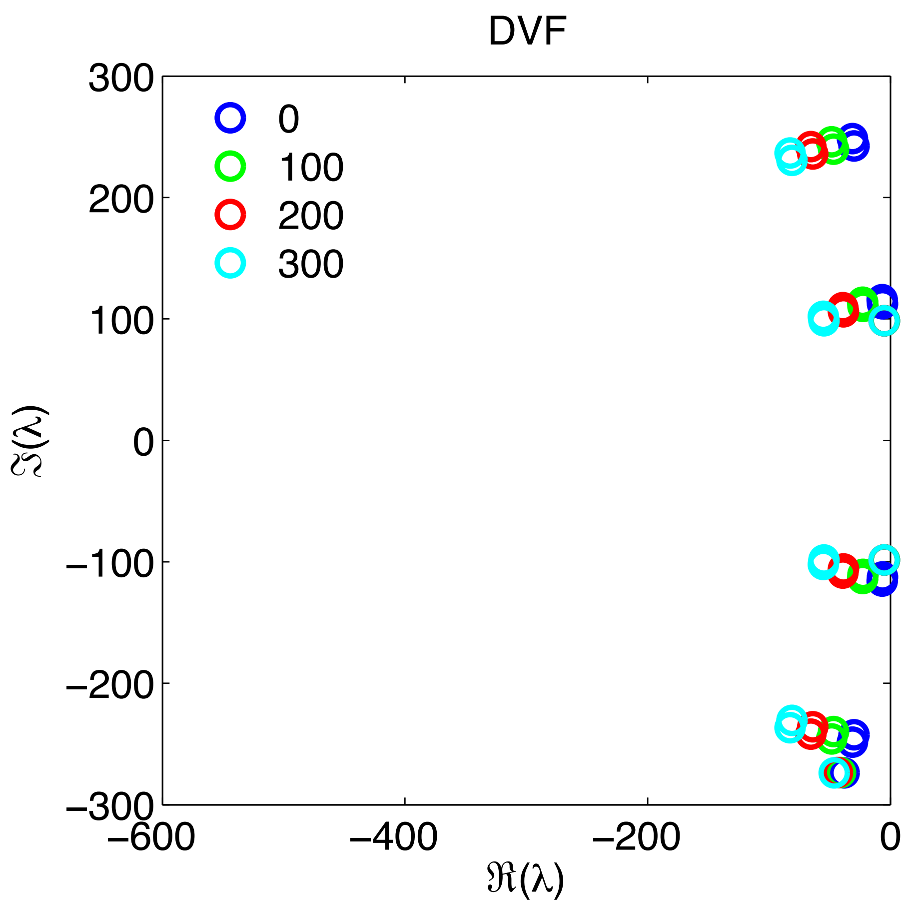
SDF and DVF in FSI
Analysis of a toy model

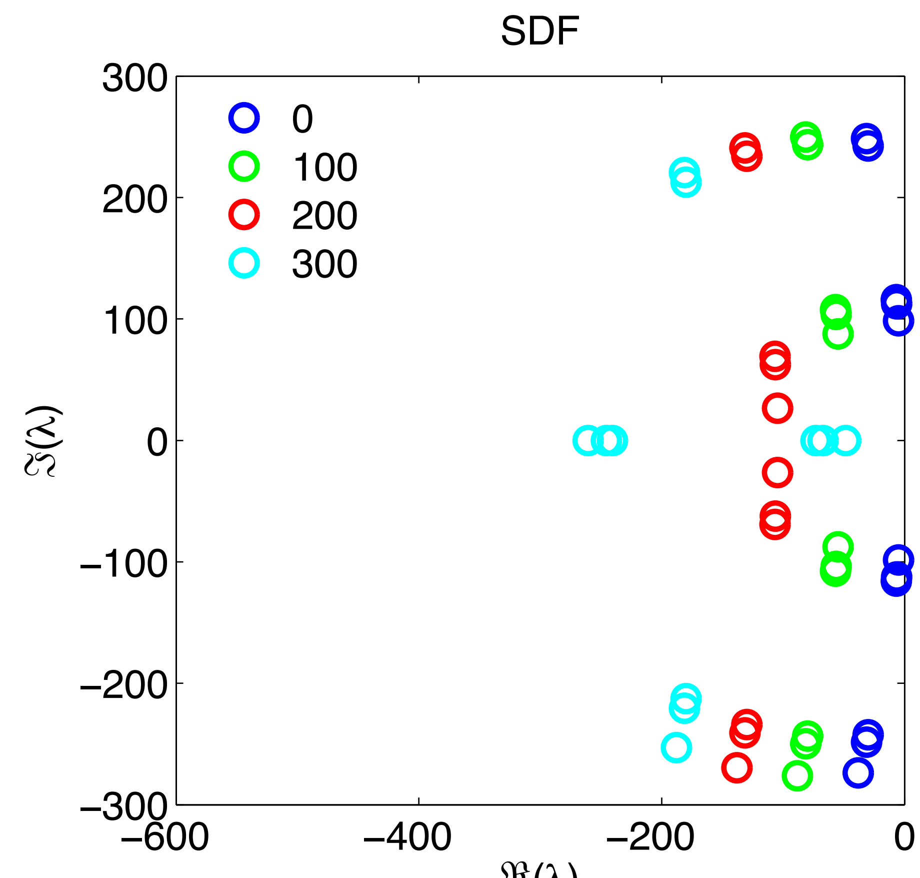
- Simplified FSI problem, with SDF or DVF

Added mass (FSI)

$$\begin{cases} (M_s + M_A)\dfrac{\mathrm{d}\hat{\boldsymbol{v}}}{\mathrm{d}t} + K_s\hat{\boldsymbol{d}} &= R + \gamma_v H_v^T M_\Gamma (Z_v - H_v(\hat{\boldsymbol{v}})) \\ K_\mu \dfrac{\mathrm{d}\hat{\boldsymbol{d}}}{\mathrm{d}t} &= K_\mu \hat{\boldsymbol{v}} + \gamma_d H_d^T M_\Gamma (Z_d - H_d(\hat{\boldsymbol{d}})) \end{cases}$$

- Evolution of λ for increasing γ:





N.B.: viscoelastic structure

SDF and DVF in FSI
Analysis of a toy model

Sensitivity

- Let $(\lambda(\gamma), \Phi(\gamma))$ an eigenmode. Assuming full observation:
  - Velocity filter: $\left.\dfrac{\partial\lambda}{\partial\gamma_v}\right|_{\gamma_v=0} = -\dfrac{1-\Phi^T M_A \Phi}{2}$
  - Displacement filter: $\left.\dfrac{\partial\lambda}{\partial\gamma_d}\right|_{\gamma_d=0} = -\dfrac{1}{2}$

Remark: In blood flows $\Phi^T M_A \Phi$ is close to 1

How to improve DVF in FSI ?

➡ Change norm used to measure the discrepancy

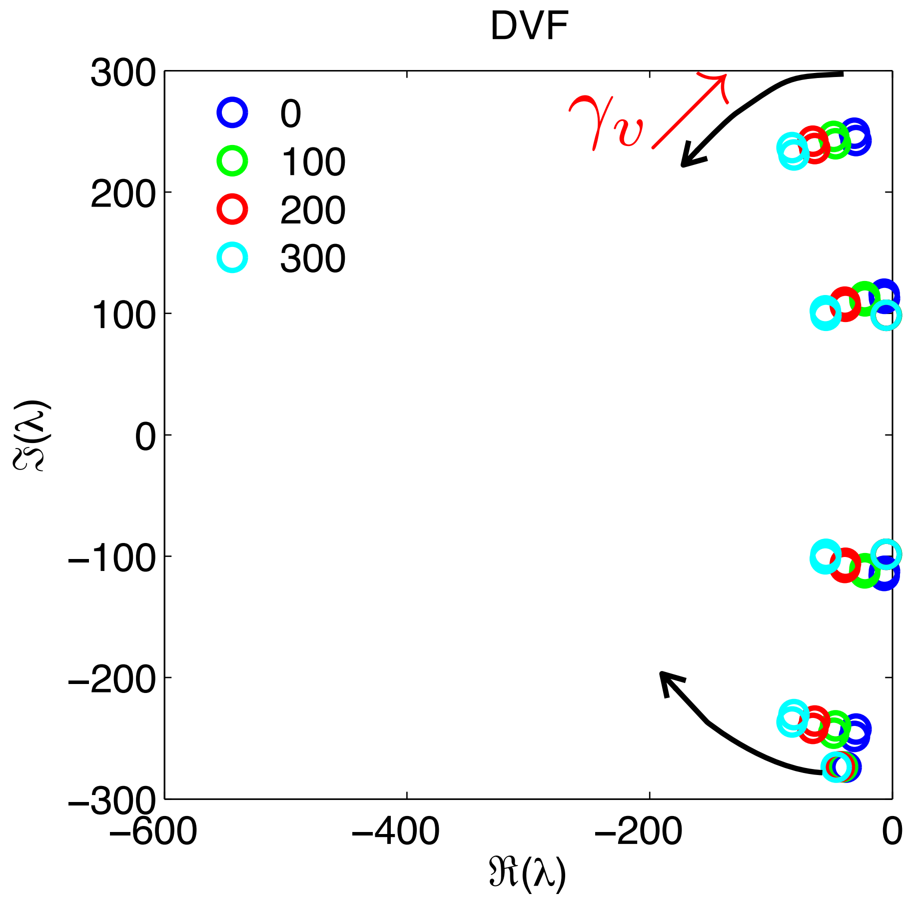
"DVFam" filter for fluid structure problems

$$\left\{ \begin{array}{rcl} M_s \dfrac{\mathrm{d}\hat{\boldsymbol{v}}}{\mathrm{d}t} + K_s \hat{\boldsymbol{d}} & = & R + \gamma_v H^T M_\Gamma (Z - H\hat{\boldsymbol{v}}) \\ \dfrac{\mathrm{d}\hat{\boldsymbol{d}}}{\mathrm{d}t} & = & \hat{\boldsymbol{v}} \end{array} \right.$$

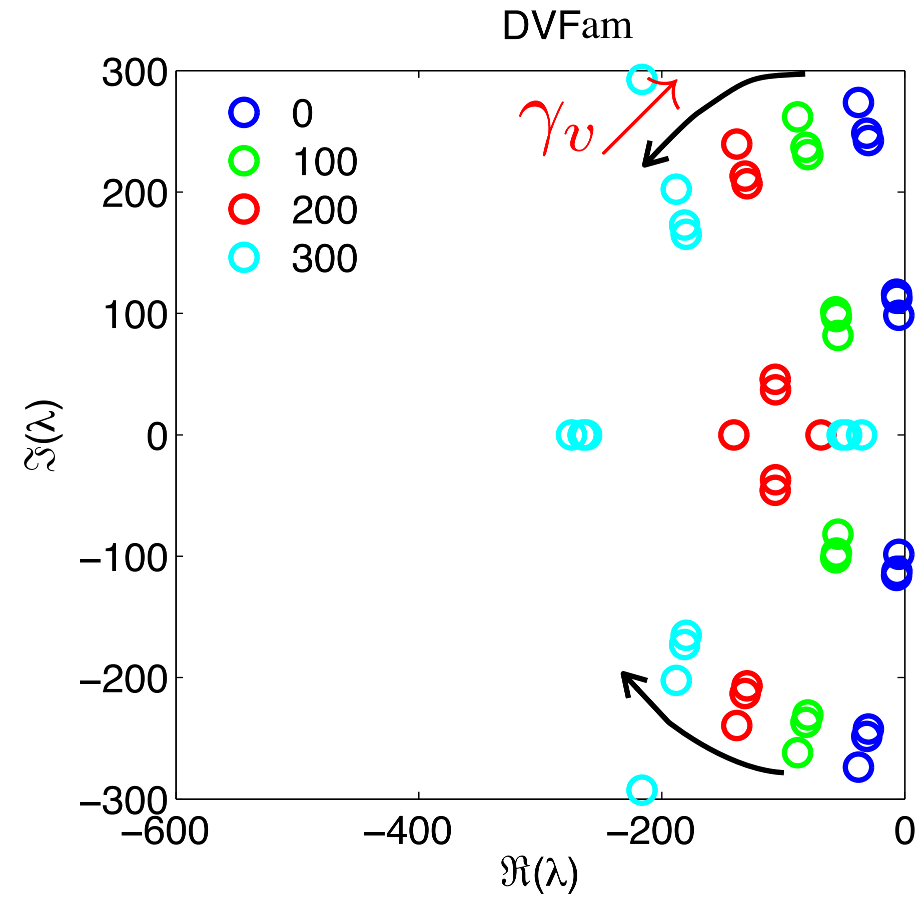
with $M_\Gamma = M_{s,\Gamma} + M_A$, where M_A is the added-mass operator.

➡ Then we recover $\left.\dfrac{\partial \lambda}{\partial \gamma_v}\right|_{\gamma_v=0} = -\dfrac{1}{2}$

Improved DVF in FSI

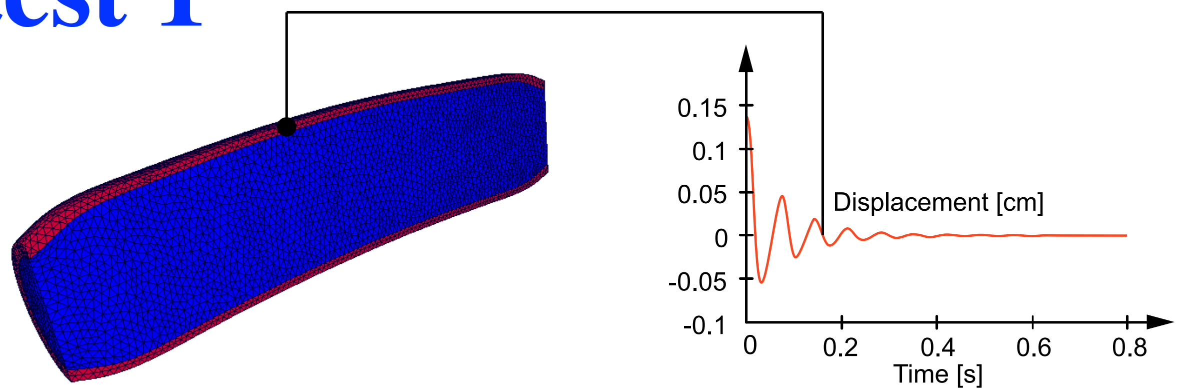


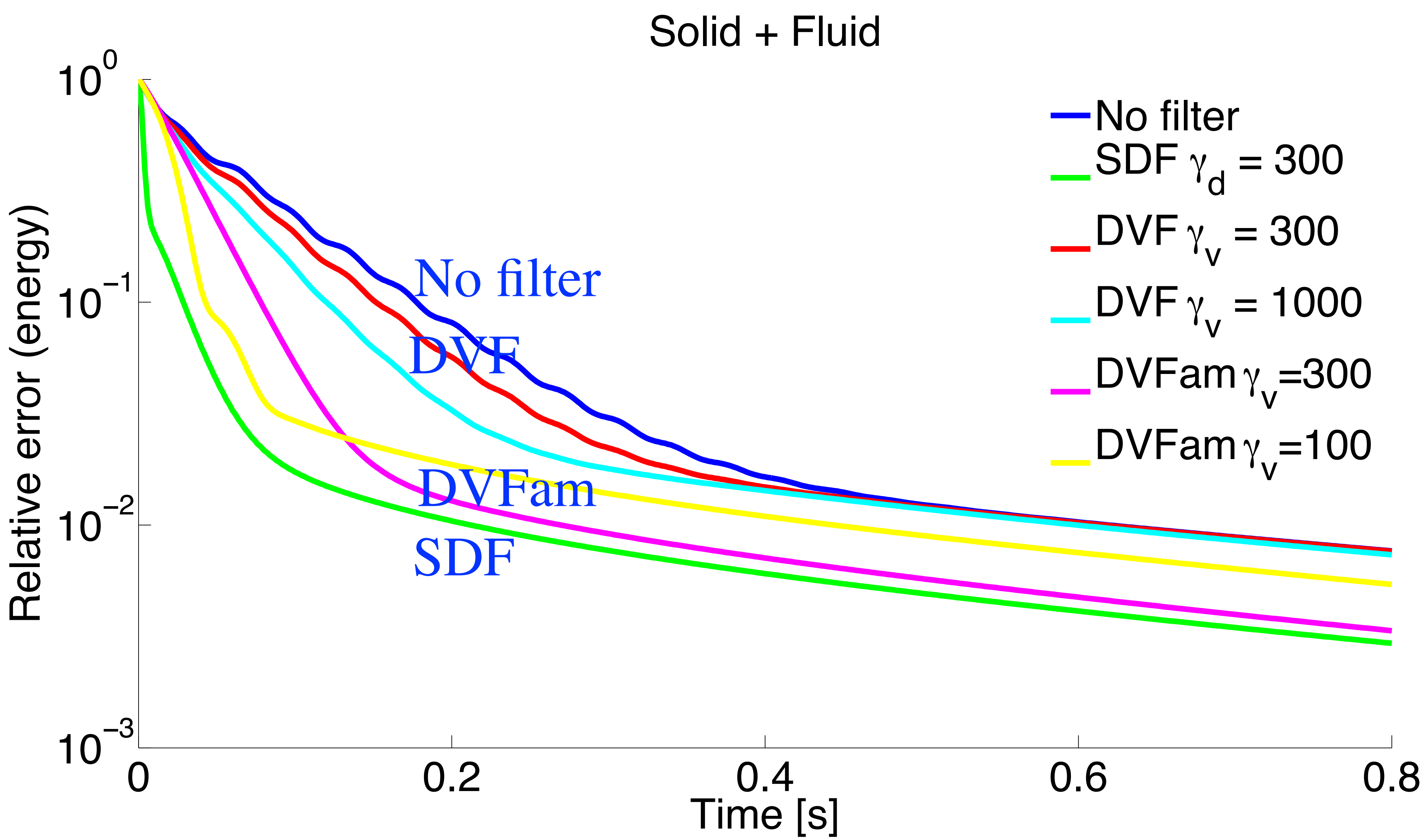
Standard DVF filter



“DVFam” filter
(norm including the added-mass)

Application to test 1





Pitfall: coupling conditions

- **Reminder SDF:** $\dfrac{\partial \boldsymbol{d}}{\partial t} = \boldsymbol{v} + \gamma_d \mathrm{Ext}(\boldsymbol{z} - \boldsymbol{d})$

  Thus $\dfrac{\partial \boldsymbol{d}}{\partial t} \neq \boldsymbol{v}$ in the solid

- At the fluid-structure interface, shall we use

$$\frac{\partial \boldsymbol{d}}{\partial t} = \boldsymbol{u} \quad \text{or} \quad \boldsymbol{v} = \boldsymbol{u} \quad ???$$

- The same analysis as before (nonlinear, spectral, sensitivity) shows that the right coupling condition is:

$$\boldsymbol{v} = \boldsymbol{u}$$

- Otherwise, it **kills** the efficiency of the SDF !

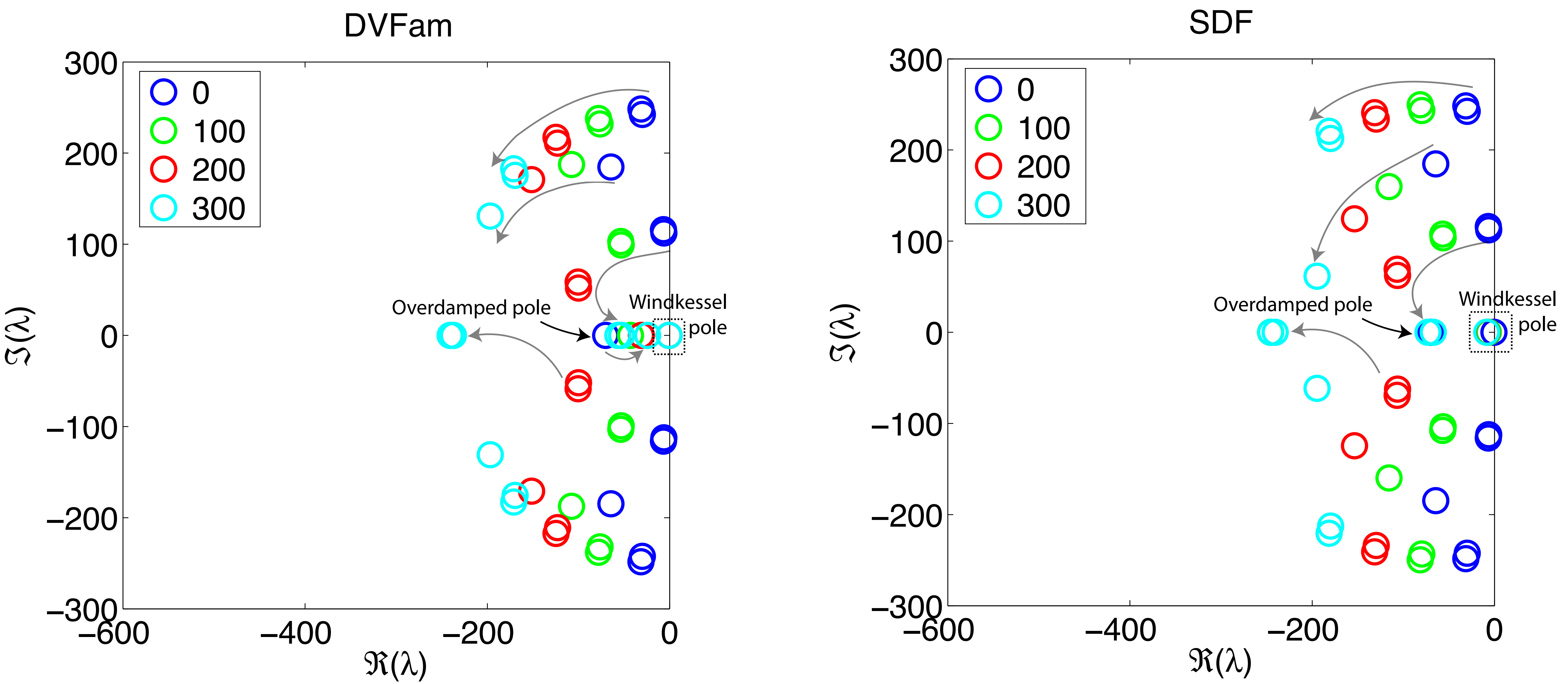
Effect of boundary conditions

Toy FSI model with a Windkessel Boundary Condition

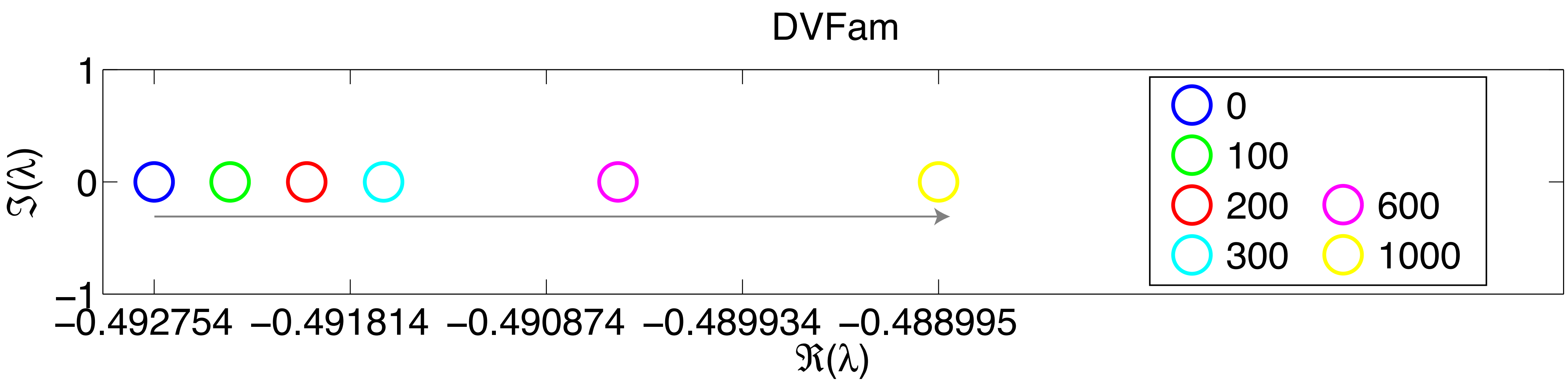
$$\begin{bmatrix} K_\mathrm{s} & 0 & 0 \\ 0 & M_\mathrm{s} + M_A & 0 \\ 0 & 0 & C \end{bmatrix} \begin{bmatrix} \dot{Y}_\mathrm{s} \\ \dot{U}_\mathrm{s} \\ \dot{\pi} \end{bmatrix} = \begin{bmatrix} 0 & K_\mathrm{s} & 0 \\ -K_\mathrm{s} & -C_\mathrm{s} - R_p S \cdot S^\intercal & S \\ 0 & -S^\intercal & -\frac{1}{R_d} \end{bmatrix} \begin{bmatrix} Y_\mathrm{s} \\ U_\mathrm{s} \\ \pi \end{bmatrix}$$

$$C \frac{d\pi}{dt} + \frac{\pi}{R_d} = Q$$

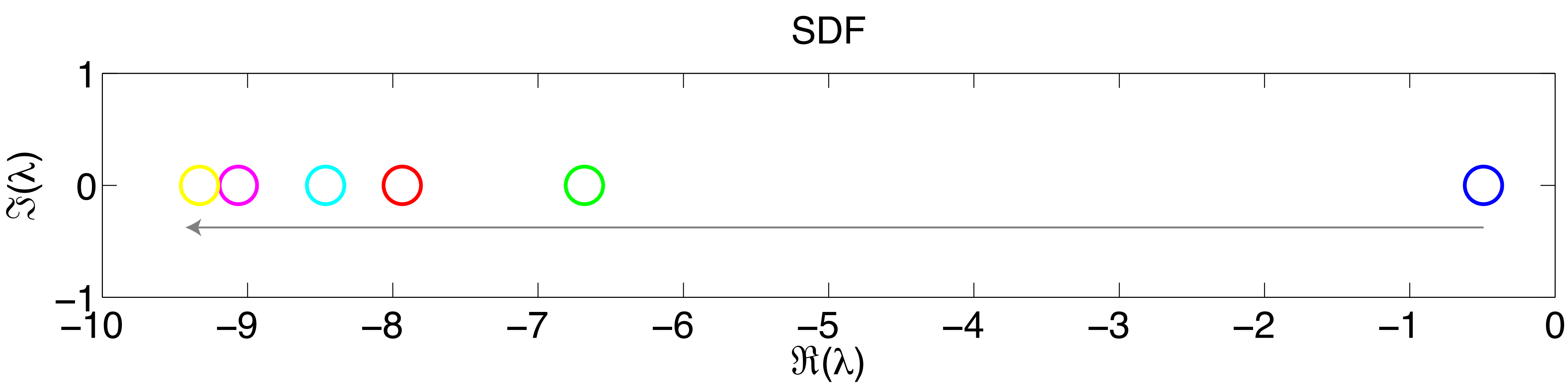
- Analytical sensitivity analysis still possible
- It confirms the observations of the numerical spectral analysis



DVFam
SDF
0
100
200
300
Overdamped pole
Windkessel pole
300
200
100
0
−100
−200
−300
−600
−400
−200
0
ℜ(λ)
ℑ(λ)



➡ DVFam (slightly) **destabilizes** the Windkessel pole



➡ SDF reasonnably good at improving the Windkessel pole

Effect of fluid dissipation

In the toy FSI model, replace the potential fluid by Stokes:

$$\begin{cases} \rho_{\mathrm{f}} \partial_t \boldsymbol{u}_{\mathrm{f}} - \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_{\mathrm{f}}(\boldsymbol{u}_{\mathrm{f}}, p) = \boldsymbol{0}, & \text{in } \Omega_0^{\mathrm{f}} \\ \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{f}} = 0, & \text{in } \Omega_0^{\mathrm{f}} \end{cases}$$

- First 100 smallest eigenvalues in module :
  - Real
  - Almost the same with Stokes or with Stokes + Structure
  - Almost unaffected by any filter

Summary

	Displacement meas.		Velocity meas.	
	SDF with $u=v$	SDF with $u=\dot{d}$	DVF	DVFam
Added-mass				
Dissipative BC				
Fluid viscosity				

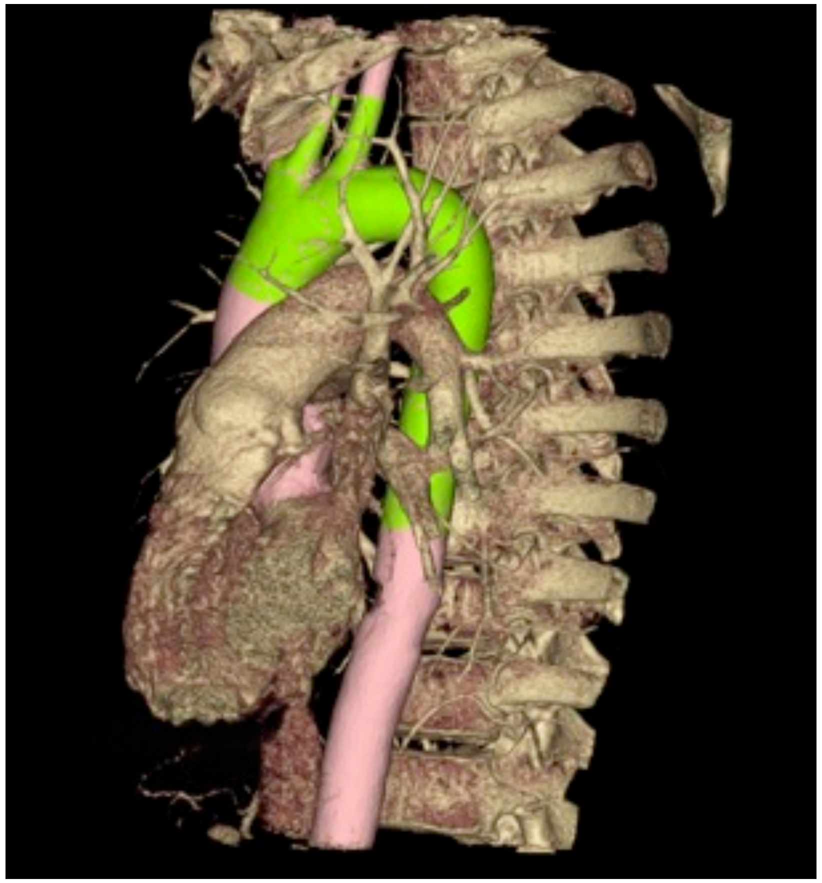
Possible remedies

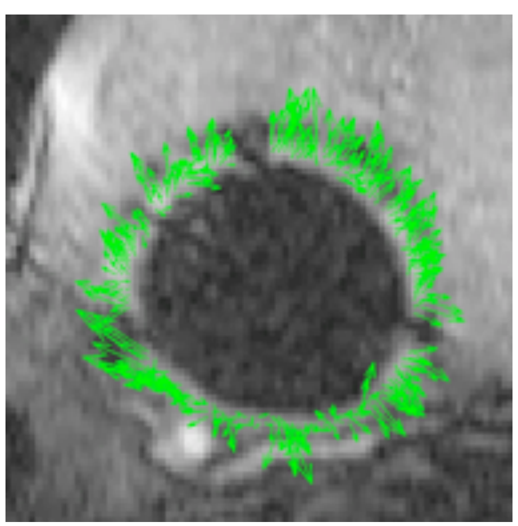
- Add pressure measurements and consider Windkessel observer like:

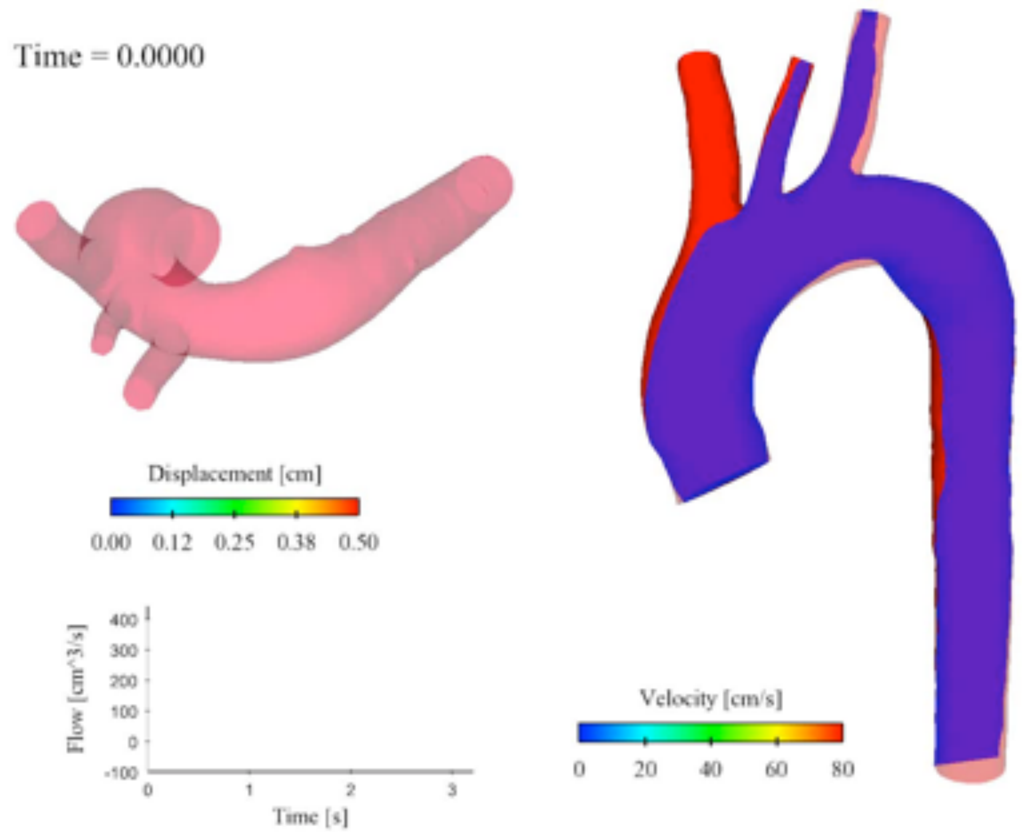
$$R_{\mathrm{d}} C \dot{\hat{\pi}} + \hat{\pi} = R_{\mathrm{d}} \hat{Q} + \gamma_{\pi}(z_{\pi} - \hat{\pi}),$$

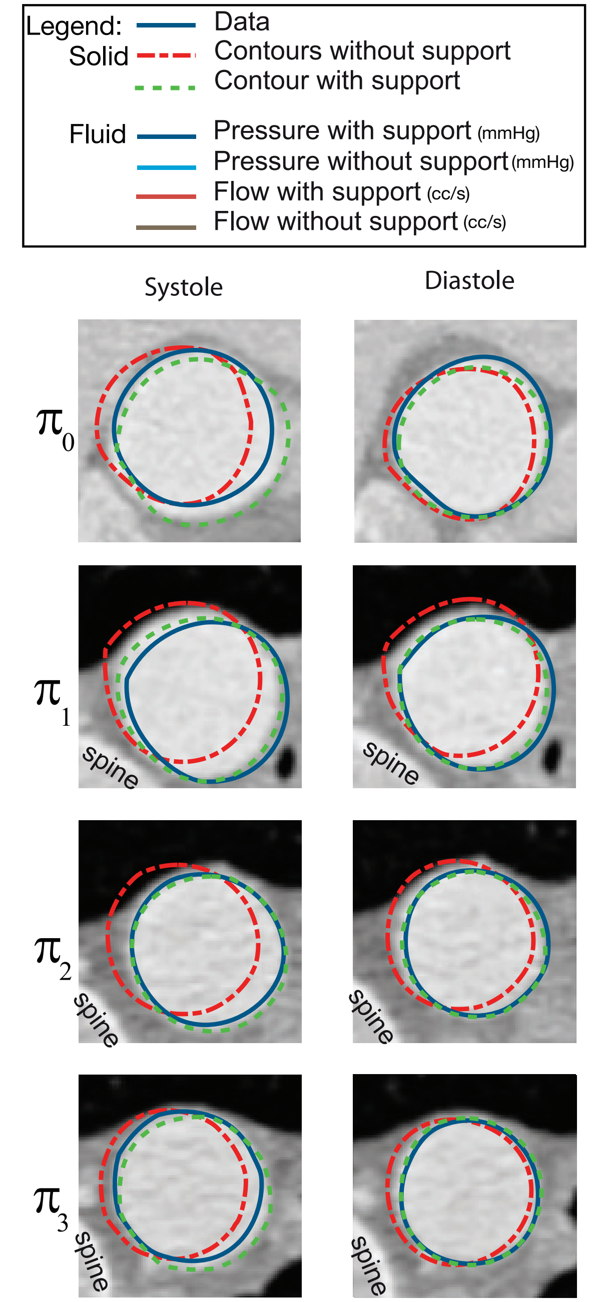
- Add fluid measurements and devise a filter for the fluid

Application: external tissue estimation









$$\boldsymbol{\sigma}_s \boldsymbol{n} = -k_s \boldsymbol{d} - c_s \frac{\partial \boldsymbol{d}}{\partial t}$$

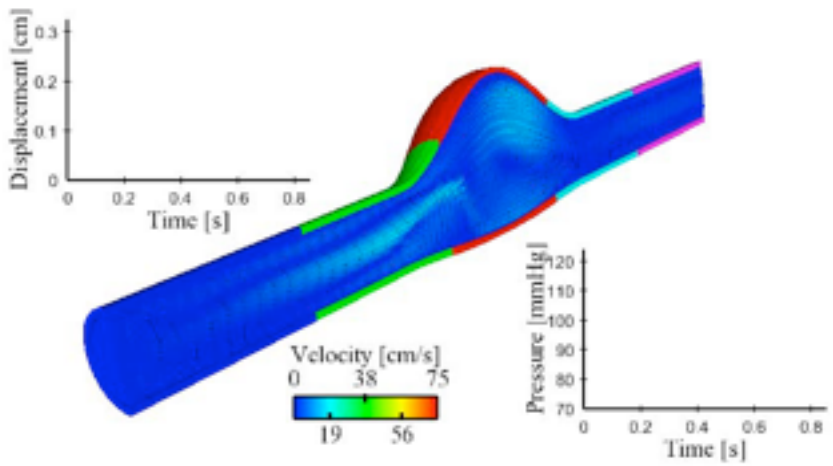
with heterogeneous coefficients

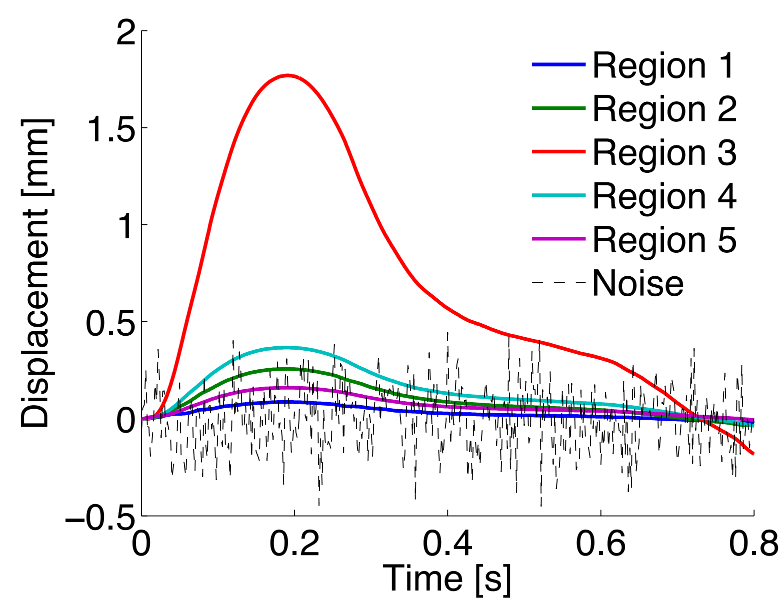
Moireau, Xiao, Astorino, Figueroa, Chapelle, Taylor, JFG, (BMMB 2012)

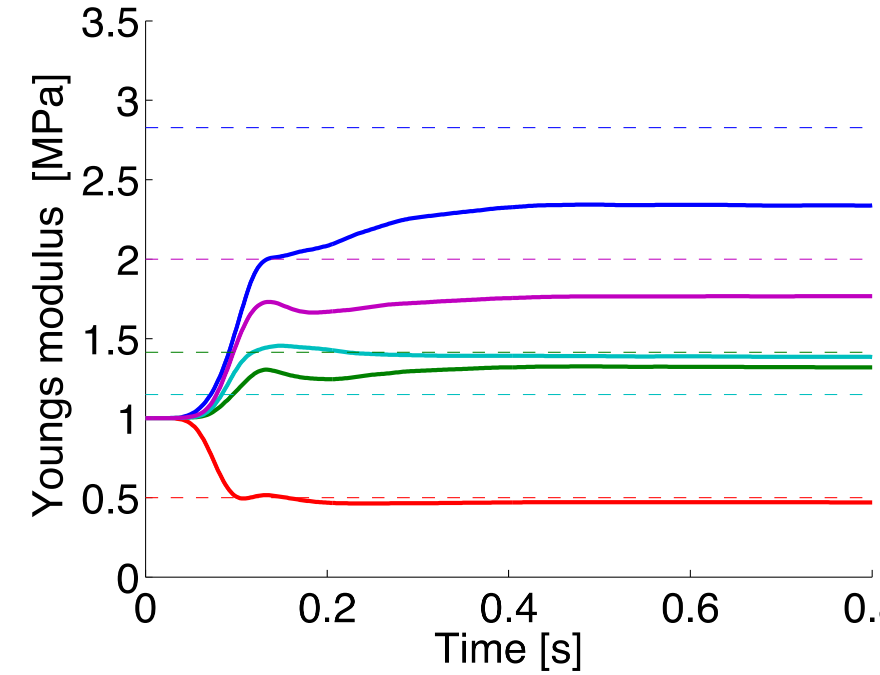
Moireau, Bertoglio, Xiao, Figueroa, Taylor, Chapelle, JFG, (BMMB 2013)

Application: arterial stiffness estimation

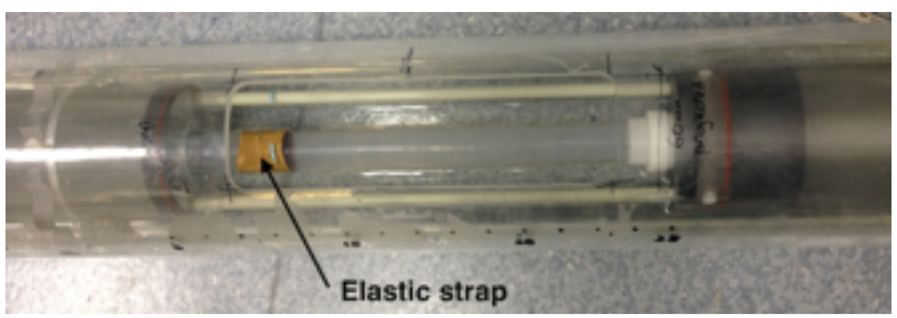
Synthetic data

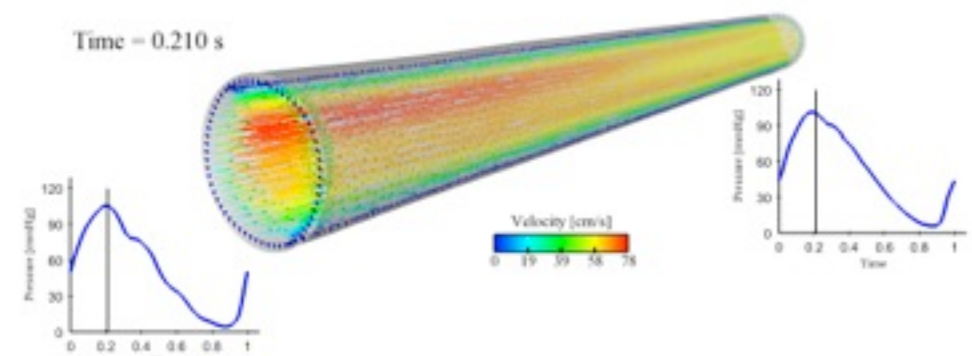




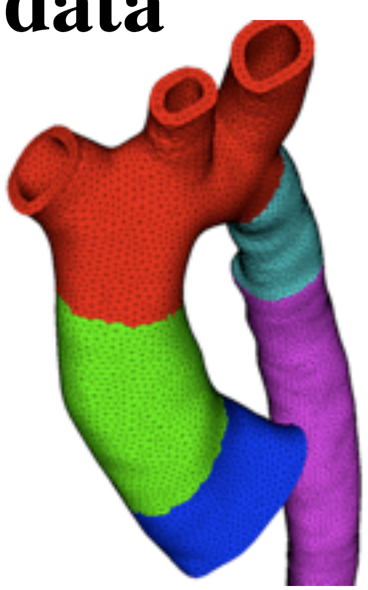


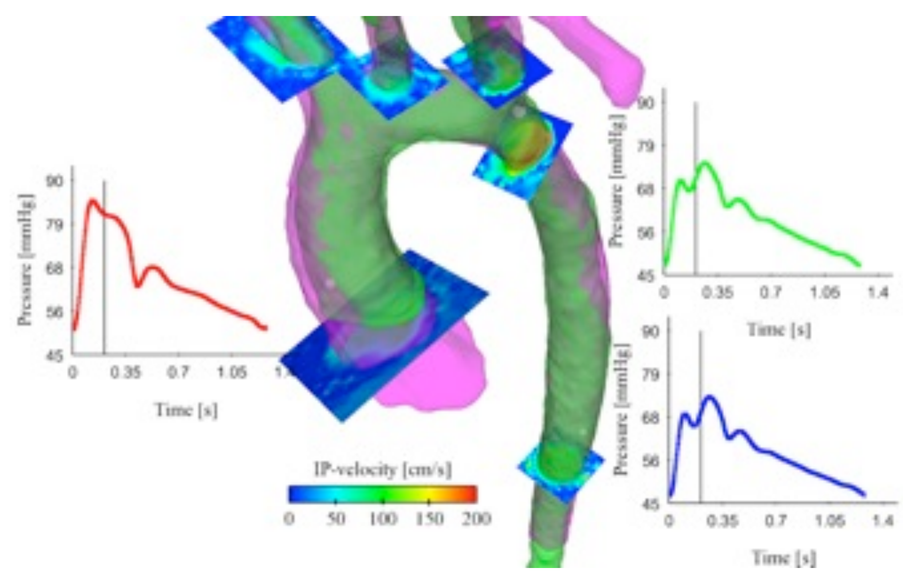
Experimental data (KCL & Sheffield, euHeart)

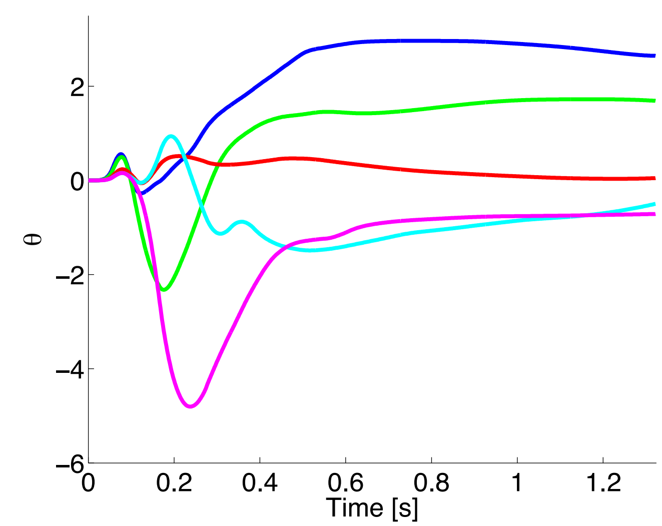




Clinical data







Bertoglio et al., J. Biomech *2014*