Added mass effect in forward and inverse fluid-structure interaction algorithms

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Jean-Frédéric Gerbeau

INRIA Paris & Sorbonne Universités UPMC
France
Fluid-structure interaction in blood flows

\[ \rho^f \left( \frac{\partial u}{\partial t} + (u - w) \cdot \nabla u \right) - 2\mu \text{div}\epsilon(u) + \nabla p = 0, \quad \text{in} \quad \Omega^f(t) \]

\[ \text{div} u = 0, \quad \text{in} \quad \Omega^f(t) \]

\[ \rho^s \frac{\partial^2 d}{\partial t^2} - \text{div}(F(d)S(d)) = 0, \quad \text{in} \quad \hat{\Omega}^s \]
Possible application: avoid clinical exams?

- **Example:** aortic coarctation
- After surgical repair, patients must be followed on a regular basis
- Exercise test is often necessary to assess the patient condition
- **Question:** With computer simulations, can we extrapolate the rest test to avoid the stress test?
- Maybe... if we are able to “personalize” an FSI model of the aorta

In collaboration with R. Hose (Sheffield), I. Valverde, P. Beerbaum (KCL)
Outline

- Forward problem in Fluid-Structure Interaction
- Inverse problem in Fluid-Structure Interaction
Fluid-Structure coupling

- **Partitioned approach:**

  - **Explicit** scheme: one iteration Fluid/Structure at each time step
  - **Implicit** scheme: many Fluid/Structure subiterations at each time step
Explicit coupling: some observations

- Explicit algorithms are a priori very efficient:
  \[
  \text{FSI cost} \approx \text{FLUID cost} + \text{SOLID cost}
  \]
- ... but naive Dirichlet-Neumann iterations are unstable !
- Explicit coupling is stable and widely used in aeroelasticity !
- Empirical observations for explicit coupling in blood flows:
  - Instabilities disappear when the solid density is (artificially) increased
  - Instabilities are independent of the time step
  - The instability is sensitive to the length of the domain
Implicit / Explicit coupling

Two approaches:

- Improve implicit iterations (Fixed point, Newton, ...)

- Devise explicit coupling algorithms:
A 2D simplified model

(Causin, JFG, Nobile, 2005)

- **Solid**: string model (small displacements)
  \[ \rho^s \varepsilon \dddot{d} + Ld = p_{|\Sigma}, \quad {\text{in}} \quad \Sigma, \]
  with
  - \(d\): vertical displacement
  - \(\varepsilon\): vessel thickness
  - \(L\): linear operator (for instance \(L\eta = a\eta - b\frac{\partial^2 \eta}{\partial x^2}\))
A 2D simplified model

(Causin, JFG, Nobile, 2005)

- **Solid**: string model (infinitesimal displacements)
  \[ \rho^s \varepsilon \ddot{d} + Ld = p|_{\Sigma}, \quad \text{in} \quad \Sigma, \]

- **Fluid**: fixed fluid domain, no viscous/convective terms
  \[
  \begin{align*}
  & \rho^f \frac{\partial u}{\partial t} + \nabla p = 0, \quad \text{in} \quad \Omega^f \\
  & \text{div } u = 0, \quad \text{in} \quad \Omega^f \\
  & u \cdot n = \dot{d}, \quad \text{on} \quad \Sigma \\
  & u \cdot n = 0, \quad \text{on} \quad \Gamma_1 \\
  & p = 0, \quad \text{on} \quad \Gamma_2
  \end{align*}
  \]

\[
  \begin{align*}
  & -\Delta p = 0, \quad \text{in} \quad \Omega^f \\
  & \frac{\partial p}{\partial n} = -\rho^f \frac{\partial u}{\partial t} \cdot n = -\rho^f \ddot{d}, \quad \text{on} \quad \Sigma \\
  & \frac{\partial p}{\partial n} = 0, \quad \text{on} \quad \Gamma_1 \\
  & p = 0 \quad \text{on} \quad \Gamma_2
  \end{align*}
  \]

- **Physics**: reproduces propagation phenomena
- **Numerics**: explicit coupling unstable
The added-mass operator

\[
\begin{align*}
\text{Fluid:} & \quad \left\{ \begin{array}{ll}
-\Delta p = 0, & \text{in } \Omega^f \\
\frac{\partial p}{\partial n} = -\rho_f \ddot{d}, & \text{on } \Sigma \\
\frac{\partial p}{\partial n} = 0, & \text{on } \Gamma_1 \\
p = 0 & \text{on } \Gamma_2
\end{array} \right. \\
\text{Solid:} & \quad \rho_s \varepsilon \ddot{d} + Ld = p|_\Sigma, \text{ in } \Sigma,
\end{align*}
\]

Steklov-Poincaré operator

The operator \( \mathcal{M}_A : H^{-\frac{1}{2}}(\Sigma) \to H^{\frac{1}{2}}(\Sigma) \) defined as: for each \( g \in H^{-\frac{1}{2}}(\Sigma) \) we set \( \mathcal{M}_A(g) \overset{\text{def}}{=} q|_{\Gamma_w}, \) where \( q \in H^1(\Omega^f) \) solves

\[
\begin{align*}
-\Delta q &= 0, & \text{in } \Omega^f \\
\frac{\partial q}{\partial n} &= g, & \text{on } \Sigma \\
\frac{\partial q}{\partial n} &= 0, & \text{on } \Gamma_1 \\
q &= 0, & \text{on } \Gamma_2
\end{align*}
\]

is a linear, compact, positive and self-adjoint operator in \( L^2(\Sigma). \)

From this definition, we have

\[
p|_\Sigma = \mathcal{M}_A (-\rho_f \ddot{d}) = -\rho_f \mathcal{M}_A \ddot{d}
\]
The added-mass effect

\[
\begin{align*}
-\Delta p &= 0, \quad \text{in } \Omega^f \\
\frac{\partial p}{\partial n} &= -\rho^f \ddot{d}, \quad \text{on } \Sigma \\
\frac{\partial p}{\partial n} &= 0, \quad \text{on } \Gamma_1 \\
p &= 0 \quad \text{on } \Gamma_2
\end{align*}
\]

Fluid:

\[
\begin{align*}
\rho^s \varepsilon \ddot{d} + Ld &= p|\Sigma, \quad \text{in } \Sigma, \quad (1) \\
p|\Sigma &= -\rho^f \mathcal{M}_A \ddot{d}
\end{align*}
\]

Solid:

\[
(\rho^s \varepsilon + \rho^f \mathcal{M}_A) \ddot{d} + Ld = 0, \quad \text{in } \Sigma, \quad (2)
\]

Remarks:

What kind of time integration scheme of (2) arises from the explicit coupling of (1)?
Explicit coupling and added-mass

\[
\begin{align*}
\text{Fluid:} & \quad \left\{ \begin{array}{l}
\rho^f \frac{u^{n+1} - u^n}{\delta t} + \nabla p^{n+1} = 0 \\
\text{div } u^{n+1} = 0 \\
u^{n+1} \cdot n = \frac{d^n - d^{n-1}}{\delta t}
\end{array} \right. \\
\implies & \quad \left\{ \begin{array}{l}
-\Delta p^{n+1} = 0 \\
\frac{\partial p^{n+1}}{\partial n} = -\rho^f \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2}
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\text{Solid:} & \quad \rho^s \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + Ld^n = p^{n+1}_{|\Sigma}
\end{align*}
\]

Condensed FSI problem:

\[
\begin{align*}
\rho^s \varepsilon \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} + \rho^f \mathcal{M}_A \frac{d^n - 2d^{n-1} + d^{n-2}}{\delta t^2} + Ld^n &= 0
\end{align*}
\]

Explicit coupling yields an explicit discretization of the added mass
An unconditional instability result

Proposition (Causin-JFG-Nobile 05)

Let $\lambda_{\text{max}}$ be the largest eigenvalue of $\mathcal{M}_A$ and assume that $L\eta = a\eta$. Then, the previous explicit coupling scheme is unconditionally unstable whenever

$$\frac{\rho^f \lambda_{\text{max}}}{\rho^s \varepsilon} \geq 1.$$  (1)

- The instability condition confirms the empirical observations:
  - Instabilities depend on the density ratio
  - The instability condition does not depend on the time step
  - Instabilities occur when the structure is thin and slender (higher $\lambda_{\text{max}}$)

- Other time schemes have been considered by Förster-Wall-Ramm 07 with analogous conclusions

- Do not forget that the first assumption to build this toy model was incompressibility
Three ideas:

- Treat implicitly the added-mass effect (incompressibility, pressure stress)
- Treat explicitly the fluid domain motion, convective and viscous effects
- Perform this using a projection scheme (Chorin-Teman) within the fluid

*(Fernández, JFG, Grandmont, 2007)*
The Chorin-Teman projection scheme

- Incompressible Navier-Stokes equations:
  \[
  \rho^f \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) - 2\mu \text{div} \epsilon(u) + \nabla p = 0, \quad \text{in} \quad \Omega^f
  \]
  \[
  \text{div} u = 0, \quad \text{in} \quad \Omega^f
  \]

- Viscous step:
  \[
  \begin{cases}
  \rho^f \left( \frac{\tilde{u}^{n+1} - u^n}{\delta t} + \tilde{u}^{n+1} \cdot \nabla \tilde{u}^{n+1} \right) - 2\mu \text{div} \epsilon(\tilde{u}^{n+1}) = 0, \quad \text{in} \quad \Omega \\
  \tilde{u}^{n+1} = 0, \quad \text{on} \quad \partial\Omega
  \end{cases}
  \]

- Projection step:
  \[
  \begin{cases}
  \rho^f \frac{u^{n+1} - \tilde{u}^{n+1}}{\delta t} + \nabla p^{n+1} = 0, \quad \text{in} \quad \Omega \\
  \text{div} u^{n+1} = 0, \quad \text{in} \quad \Omega \\
  u^{n+1} \cdot n = 0, \quad \text{on} \quad \partial\Omega
  \end{cases} \Rightarrow \begin{cases}
  -\Delta p^{n+1} = -\frac{\rho^f}{\delta t} \text{div} \tilde{u}^{n+1}, \quad \text{in} \quad \Omega \\
  \frac{\partial p^{n+1}}{\partial n} = 0, \quad \text{on} \quad \partial\Omega
  \end{cases}
  \]
Semi-implicit coupling: explicit part

- Viscous sub-step:

\[ d^{f,n+1} = \text{Ext}(d^n_{|\Sigma}), \quad w^{n+1} = \frac{d^{f,n+1} - d^n}{\delta t}, \quad \Omega^{f,n+1} = (I + d^{f,n+1})(\Omega^f), \]

\[ \rho^f \left( \frac{\tilde{u}^{n+1} - u^n}{\delta t} + (\tilde{u}^{n+1} - w^{n+1}) \cdot \nabla \tilde{u}^{n+1} \right) - 2\mu \text{div} \varepsilon(\tilde{u}^{n+1}) = 0, \quad \text{in} \quad \Omega^{f,n+1} \]

\[ \tilde{u}^{n+1} = w^{n+1}, \quad \text{on} \quad \Sigma^{n+1} \]

- Fluid domain, viscous and convective effects explicitly treated
Semi-implicit coupling: implicit part

- Fluid projection sub-step (in a known domain):
  \[
  \begin{aligned}
  \frac{\rho_f}{\delta t} u^{n+1} - \tilde{u}^{n+1} + \nabla p^{n+1} &= 0, \quad \text{in } \Omega_{f,n+1} \\
  \text{div} u^{n+1} &= 0, \quad \text{in } \Omega_{f,n+1} \\
  u^{n+1} \cdot n &= \frac{d^{n+1} - d^n}{\delta t} \cdot n, \quad \text{on } \Sigma^{n+1}
  \end{aligned}
  \]

- Solid equation:
  \[
  \begin{aligned}
  \rho_s \frac{d^{n+1} - 2d^n + d^{n-1}}{\delta t^2} - \text{div} \left( F(d^{n+1}) S(d^{n+1}) \right) &= 0, \quad \text{in } \hat{\Omega}^s \\
  F(d^{n+1}) S(d^{n+1}) \hat{n} &= J(d_{f,n+1}) \sigma(\tilde{u}^{n+1}, p^{n+1}) F(d_{f,n+1})^{-T} \hat{n}, \quad \text{on } \hat{\Sigma}
  \end{aligned}
  \]

- Projection sub-step in a fixed fluid domain
- Implicit part solved with much cheaper inner iterations
A stability result (linear case)

Proposition: *(Fernandez-JFG-Grandmont 2007)*

Assume the interface matching operator to be $L^2$-stable. Then, under condition

$$\rho^s \geq C \left( \rho^f \frac{h}{H^\alpha} + 2 \frac{\mu \delta t}{hH^\alpha} \right),$$

with $\alpha \overset{\text{def}}{=} \begin{cases} 0, & \text{if } \overline{\Omega}^s = \Sigma, \\ 1, & \text{if } \overline{\Omega}^s \neq \Sigma, \end{cases}$

the following discrete energy inequality holds:

$$\frac{1}{\delta t} \left[ \frac{\rho^f}{2} \| u_h^{n+1} \|_{0,\Omega^f}^2 - \frac{\rho^f}{2} \| u_h^n \|_{0,\Omega^f}^2 + \frac{\rho^s}{2} \left\| \frac{d_H^{n+1} - d_H^n}{\delta t} \right\|_{0,\Omega^f}^2 - \frac{\rho^s}{2} \left\| \frac{d_H^n - d_H^{n-1}}{\delta t} \right\|_{0,\Omega^f}^2 \right]$$

$$+ \frac{1}{2\delta t} \left[ a^s(d_H^{n+1}, d_H^{n+1}) - a^s(d_H^n, d_H^n) \right] + \mu \| \epsilon(u_h^{n+1}) \|_{0,\Omega^f}^2 \leq 0$$

Therefore, the semi-implicit coupling scheme is conditionnally stable in the energy norm.
Navier-Stokes / nonlinear shell coupling

- Straight cylinder: 50 time steps of length $\delta t = 2 \times 10^{-4} s$

<table>
<thead>
<tr>
<th>COUPLING</th>
<th>ALGORITHM</th>
<th>CPU time</th>
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</thead>
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<tr>
<td>Implicit</td>
<td>FP-Aitken</td>
<td>24.86</td>
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<tr>
<td></td>
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<td>6.05</td>
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<tr>
<td></td>
<td>Newton</td>
<td>4.77</td>
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<tr>
<td>Semi-Implicit</td>
<td>Newton</td>
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</tbody>
</table>

2001
2003
2007
Navier-Sokes / Nonlinear shell coupling

- Carotid artery (in-vivo model): 9 cardiac cycles, 4500 times steps
  - $\delta t = 1.68 \times 10^{-3}\,s$

- Fluid: 70047 Tetrahedra ($P_1/P_1$ FE)

- Solid: 8103 Quadrilaterals (MITC4 FE)

- Parameters: $\mu = 0.035\,poise$, $\rho^f = 1\,g/cm^3$, $\rho^s = 1.2\,g/cm^3$, $E = 6 \times 10^6\,dynes/cm^2$, $\nu = 0.3$.

<table>
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</tr>
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<tbody>
<tr>
<td>Implicit</td>
<td>6.7</td>
</tr>
<tr>
<td>Semi-Implicit</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Dimensionless CPU time
Recent approaches: explicit schemes

- Idea: only solid inertia needs to be implicitly coupled to the fluid

- Fluid

\[
\begin{cases}
\rho^f \partial_t u - \text{div} \sigma(u, p) = 0 & \text{in } \Omega^f \\
\text{div} u = 0 & \text{in } \Omega^f \\
\quad u = \dot{d} & \text{on } \Sigma
\end{cases}
\]

- Thin solid

\[
\begin{cases}
\rho^s \epsilon \partial_t \dot{d} + L^e d = -\sigma(u, p)n & \text{on } \Sigma \\
\dot{d} = \partial_t d & \text{on } \Sigma
\end{cases}
\]

\[
\sigma(u^n, p^n)n + \frac{\rho^s \epsilon}{\tau} u = \frac{\rho^s \epsilon}{\tau} d^{n-1} - L^e d^* \quad \text{on } \Sigma, \quad d^* = \begin{cases}
0 \\
\quad d^{n-1} \\
\quad d^{n-1} + \tau \dot{d}^{n-1}
\end{cases}
\]

- Added-mass free and parameter free
- Key issue is now the accuracy!  

Fernández 2012
Outline

• Forward problem in Fluid-Structure Interaction
• Inverse problem in Fluid-Structure Interaction
Medical data assimilation

Models
- Navier-Stokes equations
- Solid mechanics
- ...

Measurements
- medical imaging (CT, MRI, ...)
- blood flow (US, PC-MRI,...)
- pressure (catheter)
- ...

Estimate parameters
- artery wall stiffness
- boundary condition
- ...

Access to hidden quantities
- pressure
- wall stress

Improve measurements
- regularization
- interpolation
- ...
Introduction to Luenberger observers

- Dynamical system:
  \[
  \begin{aligned}
  \frac{dX}{dt} &= A(X, \theta) \\
  X(0) &= X_0 
  \end{aligned}
  \]

- Example of state variable: \( X = [u, d, v] \)

- Example of parameters: \( \theta = [\text{Young modulus}, \text{boundary conditions}, \ldots] \)

Imperfect knowledge of \( X(t = 0) \) and \( \theta \): \( \hat{X}_0 \) and \( \hat{\theta}_0 \)
Introduction to Luenberger observers

- Partial observations of $X$: $Z = H(X)$

Data: I. Valverde, P. Beerbaum (euHeart project).

- Minimize

$$J(X_0, \theta) = \frac{1}{2} \int_0^T \|Z - H(X(t))\|_W^2 \, dt + \frac{1}{2} \|X_0 - \hat{X}_0\|_P^2 + \frac{1}{2} \|\theta - \hat{\theta}_0\|_P^2$$

where $X(t)$ is the solution of the state equation associated to $(X_0, \theta)$. 

State estimation  Parameters identification
Data assimilation

• Variational approach:
  - Optimization algorithms
  - Usually based on gradient (adjoint equations)

In hemodynamics: Piccinelli, Mirabella, Passerini, Haber, Veneziani, 2012
               D’Elia, Perego, Veneziani, 2012
               Bertoglio, Chapelle, Fernandez, JFG, Moireau, 2013
               Bertoglio, Moireau, JFG, 2013

• Filtering approach:
  - Sequential correction of the state and the parameters

In hemodynamics: Moireau, Bertoglio, Xiao, Figueroa, Taylor, Chapelle, JFG, 2012
                 Bertoglio, Chapelle, Fernandez, JFG, Moireau, 2013
                 Bertoglio, Moireau, JFG, 2013

Strategy: reduced filtering

• Kalman filtering (UKF) is only used for the parameters $\theta$ ($p \ll N$)
• A much cheaper filter (Luenberger) is used for the state $X$
Introduction to Luenberger observers

- In this talk: only state estimation

\[ J(X_0) = \frac{1}{2} \int_0^T \|Z - H(X(t))\|^2_W \, dt + \frac{1}{2} \|X_0 - \hat{X}_0\|^2_P \]

- In dissipative system, error in initial condition is “forgotten”....

- ... but, in view of joint state-parameter estimation, we want to forget it as quickly as possible!
Introduction to Luenberger observers

- **Sequential estimation**
  - introduce a modified system: the “observer”

\[
\begin{aligned}
\frac{d\hat{X}}{dt} &= A(\hat{X}) + G(Z - H(\hat{X})) \\
\hat{X}(0) &= \hat{X}_0
\end{aligned}
\]

- with the ultimate objective to converge to the real trajectory $X(t)$
Introduction to Luenberger observers

- Search for the filter $G$ such that the **optimality criterion** is satisfied:

$$X(t) = X_{\text{argmin} J(\cdot,t)}$$

$G$ obtained from the **Riccati** or **HJB** equations.

- Intractable for PDEs

- Cheaper alternative:
  - renounce to the optimality criterion
  - build an *ad hoc* operator $G$ to have the error decreased

- Idea introduced by Luenberger in 1963.

- Also known as “nudging” in the data assimilation community

Introduction to Luenberger observers

Luenberger filter: looks simple but...

- Sometimes, there are pitfalls
- There is room for creativity!

The case of a linear dynamics:

- “Real” dynamics (without noise):
  \[
  \frac{dX}{dt} = AX + G(Z - HX) = 0
  \]

- Observer (Luenberger):
  \[
  \frac{d\hat{X}}{dt} = A\hat{X} + G(Z - H\hat{X})
  \]

- Dynamics of the error \(e_X = X - \hat{X}:
  \[
  \frac{d e_X}{dt} = (A - GH)e_X
  \]
Introduction to Luenberger observers

- Spectral properties of the error dynamics:

\[(A - GH)\Phi_k = \lambda_k \Phi_k\]

\[\leq 0\]

**Goal:** Devise an operator \(G\) to reduce \(\max (Re(\lambda_k))\)

- Typically, to decrease the initial error by a factor \(\beta\) in a time \(T_c\):

\[\max (Re(\lambda_k)) \leq \frac{\log \beta}{T_c}\]

- Ex: to have \(\beta = 10\) in \(T_c = 0.1s\), \(\max (Re(\lambda_k)) \approx -25\)
Luenberger observers in elastodynamics

- Elastodynamics equations $X = [d, v]$

- Velocity filtering: *Direct Velocity Feedback (DVF)*
  
  *(Moireau-Chapelle-Le Tallec, 2008)*

  $$
  \begin{align*}
  M_s \frac{d\hat{v}}{dt} + K_s \hat{d} &= R + \gamma_v H^T M_H (Z - H\hat{v}) \\
  \frac{d\hat{d}}{dt} &= \hat{v}
  \end{align*}
  $$

- Equation of the error: $e_v = v - \hat{v}, e_d = d - \hat{d}$

  $$
  M_s \frac{de_v}{dt} + K_s e_d = -\gamma_v H^T M_H H e_v
  $$
A trivial example: linear oscillator

\[ m \frac{d^2 e_d}{dt^2} + k e_d = -\gamma_v \frac{de_d}{dt} \]

- If \( \beta < \omega_0 \): underdamped
- If \( \beta = \omega_0 \): critically damped
- If \( \beta > \omega_0 \): overdamped

Let \( \omega_0 = \sqrt{\frac{k}{m}} \) and \( \beta = \frac{\gamma_v}{2m} \)
Luenberger observers in elastodynamics

- If the measurements are **displacements**

- First option:

\[
\begin{align*}
M_s \frac{d\hat{v}}{dt} + K_s \hat{d} &= R + \gamma_d H^T M_H (Z - H\hat{d}) \\
\frac{d\hat{d}}{dt} &= \hat{v}
\end{align*}
\]

- **Remarks:**
  - Related to the "**Image Force Method**" used in the medical imaging community
  - Poor behavior (except for systems with very large dissipation)
Luenberger observers in elastodynamics

- Displacement filtering: *Schur Displacement Feedback (SDF)*
  
  *(Moireau-Chapelle-Le Tallec, 2009)*

\[
\begin{align*}
M_s \frac{d\hat{v}}{dt} + K_s \hat{d} &= R \\
\frac{d\hat{d}}{dt} &= \hat{v} + \gamma_d K^{-1}_\mu H^T M_H(Z - H(\hat{d}))
\end{align*}
\]

with \( K_\mu = K_s + \mu H^T M_H H. \)

- Remarks:
  - Velocity is no longer the derivative of displacement
    
    \[
    \frac{\partial d}{\partial t} = v + \gamma_d \text{Ext}(z - d)
    \]
  
    - The norm matters!
Luenberger observers in elastodynamics

Velocity feedback

Displacement feedback

DVF and SDF have a similar behavior in elastodynamics
Luenberger observers in FSI

- We limit ourselves to *solid measurements*
- We are interested in:
  - The effect of the FSI coupling
  - The effect of boundary conditions
  - The effect of fluid dissipation
1st nonlinear test: stabilization to equilibrium

- Fluid initially at rest
- Initial perturbation in the solid
2d nonlinear test: hemodynamics

\[ p = \pi + R_p Q \quad \text{with} \]

Windkessel

\[
\begin{aligned}
C \frac{d\pi}{dt} + \frac{\pi}{R_d} &= Q \\
\pi|_{t=0} &= \pi_0
\end{aligned}
\]

and

\[ Q = \int_{\Gamma_{out}} u \cdot n \, dS \]
Relative error (energy) vs Time [s]

Energy norm of the error

- Solid + Fluid + Windkessel
- No filter
- DVF
- SDF

Various filter settings:
- $\gamma_d = 100$
- $\gamma_d = 200$
- $\gamma_d = 300$
- $\gamma_v = 100$
- $\gamma_v = 200$
- $\gamma_v = 300$
SDF and DVF in FSI
Analysis of a toy model

- Simplified fluid:

\[
\begin{align*}
\rho_f \frac{\partial \mathbf{u}}{\partial t} + \nabla p &= 0, \text{ in } \Omega^f \\
\nabla \cdot \mathbf{u} &= 0, \text{ in } \Omega^f \\
\mathbf{u} \cdot \mathbf{n} &= \ddot{d}, \text{ on } \Sigma
\end{align*}
\]

\[
\begin{align*}
\begin{aligned}
-\Delta p &= 0, \text{ in } \Omega^f \\
\frac{\partial p}{\partial n} &= -\rho_f \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{n} = -\rho_f \ddot{d} \cdot \mathbf{n}, \text{ on } \Sigma
\end{aligned}
\end{align*}
\]

- Let \( \mathcal{M}_A \) be the “Neumann-to-Dirichlet” operator: \( p|_\Sigma = -\rho_f \mathcal{M}_A \ddot{d} \cdot \mathbf{n} \)

- Linear elasticity:

\[
\begin{align*}
\begin{aligned}
\rho_s \dddot{\mathbf{d}} - \nabla \cdot \sigma(\mathbf{d}) &= 0, \text{ in } \Omega^s \\
\sigma(\mathbf{d}) \cdot \mathbf{n} = p|_\Sigma \mathbf{n} &= -\rho_f \mathcal{M}_A \ddot{d} \cdot \mathbf{n} \mathbf{n}, \text{ on } \Sigma
\end{aligned}
\end{align*}
\]
SDF and DVF in FSI
Analysis of a toy model

- Simplified FSI problem, with SDF or DVF

\[
\begin{align*}
(M_s + M_A) \frac{d\hat{v}}{dt} + K_s \hat{d} &= R + \gamma_v H_v^T M_{\Gamma} (Z_v - H_v(\hat{v})) \\
K_\mu \frac{d\hat{d}}{dt} &= K_\mu \hat{v} + \gamma_d H_d^T M_{\Gamma} (Z_d - H_d(\hat{d}))
\end{align*}
\]

- Evolution of $\lambda$ for increasing $\gamma$:

N.B.: viscoelastic structure
Sensitivity

- Let \((\lambda(\gamma), \Phi(\gamma))\) an eigenmode. Assuming full observation:
  - Velocity filter: \[
  \left. \frac{\partial \lambda}{\partial \gamma_v} \right|_{\gamma_v=0} = - \frac{1 - \Phi^T M_A \Phi}{2}
  \]
  - Displacement filter: \[
  \left. \frac{\partial \lambda}{\partial \gamma_d} \right|_{\gamma_d=0} = - \frac{1}{2}
  \]

**Remark:** In blood flows \(\Phi^T M_A \Phi\) is close to 1
How to improve DVF in FSI?

- Change norm used to measure the discrepancy

"DVFam" filter for fluid structure problems

\[
\begin{align*}
M_s \frac{d\hat{v}}{dt} + K_s \hat{d} &= R + \gamma_v H^T M_\Gamma (Z - H \hat{v}) \\
\frac{d\hat{d}}{dt} &= \hat{v}
\end{align*}
\]

with \( M_\Gamma = M_{s, \Gamma} + M_A \), where \( M_A \) is the added-mass operator.

- Then we recover \( \frac{\partial \lambda}{\partial \gamma_v} \bigg|_{\gamma_v=0} = -\frac{1}{2} \)
Improved DVF in FSI

Standard DVF filter

“DVFam” filter
(norm including the added-mass)
Application to test 1

<table>
<thead>
<tr>
<th>Solid + Fluid</th>
<th>No filter</th>
<th>SDF $\gamma_d = 300$</th>
<th>DVF $\gamma_v = 300$</th>
<th>DVF $\gamma_v = 1000$</th>
<th>DVFam $\gamma_v = 300$</th>
<th>DVFam $\gamma_v = 100$</th>
</tr>
</thead>
</table>

**Graph:**
- **X-axis:** Time [s]
- **Y-axis:** Relative error (energy)
- **Legend:**
  - Blue: No filter
  - Green: SDF $\gamma_d = 300$
  - Red: DVF $\gamma_v = 300$
  - Cyan: DVF $\gamma_v = 1000$
  - Magenta: DVFam $\gamma_v = 300$
  - Yellow: DVFam $\gamma_v = 100$
Pitfall: coupling conditions

- **Reminder SDF:** \( \frac{\partial d}{\partial t} = v + \gamma d \text{Ext}(z - d) \)

  Thus \( \frac{\partial d}{\partial t} \neq v \) in the solid

- At the fluid-structure interface, shall we use

  \( \frac{\partial d}{\partial t} = u \) or \( v = u \) ???

- The same analysis as before (nonlinear, spectral, sensitivity) shows that the right coupling condition is:

  \( v = u \)

- Otherwise, it **kills** the efficiency of the SDF!
Effect of boundary conditions

Toy FSI model with a Windkessel Boundary Condition

\[
\begin{bmatrix}
K_s & 0 & 0 \\
0 & M_s + M_A & 0 \\
0 & 0 & C
\end{bmatrix}
\begin{bmatrix}
\dot{Y}_s \\
\dot{U}_s \\
\dot{\pi}
\end{bmatrix}
= \begin{bmatrix}
0 & K_s & 0 \\
-K_s & -C_s - R_p S \cdot S^T & S \\
0 & -S^T & -\frac{1}{R_d}
\end{bmatrix}
\begin{bmatrix}
Y_s \\
U_s \\
\pi
\end{bmatrix}
\]

\[C \frac{d\pi}{dt} + \frac{\pi}{R_d} = Q\]

- Analytical sensitivity analysis still possible
- It confirms the observations of the numerical spectral analysis

\[49\]
DV Fam (slightly) **destabilizes** the Windkessel pole

SDF reasonably good at improving the Windkessel pole
Effect of fluid dissipation

In the toy FSI model, replace the potential fluid by Stokes:

\[
\begin{align*}
\rho_f \partial_t \mathbf{u}_f - \nabla \cdot \mathbf{\sigma}_f(\mathbf{u}_f, p) &= 0, \quad \text{in } \Omega^f_0 \\
\nabla \cdot \mathbf{u}_f &= 0, \quad \text{in } \Omega^f_0
\end{align*}
\]

- First 100 smallest eigenvalues in module:
  - Real
  - Almost the same with Stokes or with Stokes + Structure
  - Almost unaffected by any filter
### Summary

<table>
<thead>
<tr>
<th></th>
<th>Displacement meas.</th>
<th>Velocity meas.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Added-mass</strong></td>
<td><img src="thumb" alt="Smiley" /></td>
<td><img src="thumb" alt="Sad" /></td>
</tr>
<tr>
<td><strong>Dissipative BC</strong></td>
<td><img src="thumb" alt="Smiley" /></td>
<td><img src="thumb" alt="Sad" /></td>
</tr>
<tr>
<td><strong>Fluid viscosity</strong></td>
<td><img src="thumb" alt="Sad" /></td>
<td><img src="thumb" alt="Sad" /></td>
</tr>
</tbody>
</table>

#### Possible remedies

- Add pressure measurements and consider Windkessel observer like:
  \[
  R_d C \dot{\hat{\pi}} + \hat{\pi} = R_d \dot{\hat{Q}} + \gamma_\pi (z_\pi - \hat{\pi}),
  \]
- Add fluid measurements and devise a filter for the fluid
Application: external tissue estimation

\[ \sigma_s n = -k_s d - c_s \frac{\partial d}{\partial t} \]

with heterogeneous coefficients

Moireau, Xiao, Astorino, Figueroa, Chapelle, Taylor, JFG, (BMMB 2012)
Moireau, Bertoglio, Xiao, Figueroa, Taylor, Chapelle, JFG, (BMMB 2013)
Application: arterial stiffness estimation

Synthetic data

Experimental data (KCL & Sheffield, euHeart)

Clinical data

Bertoglio et al., J. Biomech 2014