A two-layer solid/liquid mixture model for fluidized granular flows with dilatancy effects

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Outline

Viscoplastic materials are characterized by

- Nonlinear relation between stress and strain rate (non-Newtonian fluid)
- Irreversible deformations inducing dissipation

Examples of such materials : snow, mud, lava, grains in a surrounding fluid (wet sand). Behaviour can be fluid or solid

Momentum equation for an incompressible ($\operatorname{div} u = 0$) viscoplastic material :

$$\partial_t u + u \cdot \nabla u - \operatorname{div} \sigma = f,$$

where the stress σ is related to the strain rate $Du = (\nabla u + (\nabla u)^t)/2$ by a nonlinear relation

$$\sigma = -p \, \mathrm{Id} + \sigma', \qquad \sigma' = F(Du),$$

with F a monotone nonlinearity (that defines the rheology).

- For a Bingham fluid

  $$F(Du) = \kappa \frac{Du}{||Du||},$$

  for some constant κ (yield stress). Multivalued meaning :

  $$\begin{cases} \|\sigma'\| \leq \kappa, \\ Du \neq 0 \text{ implies } \sigma' = \kappa \frac{Du}{\|Du\|}. \end{cases}$$

  Where $Du = 0$, the material is solid.
- $\mu(I)$ rheology (Jop, Forterre, Pouliquen 06)

  $$\sigma' = \mu(I) p \frac{Du}{\|Du\|},$$

  with $I = 2d\|Du\|/\sqrt{p/\rho_s}$, d diameter of the grains, ρ_s density of the solid.
- No justification from microscopic (anelastic) laws, just scaling laws !

We consider a dry granular material. The only non Newtonian effect that we take into account is dilatancy.

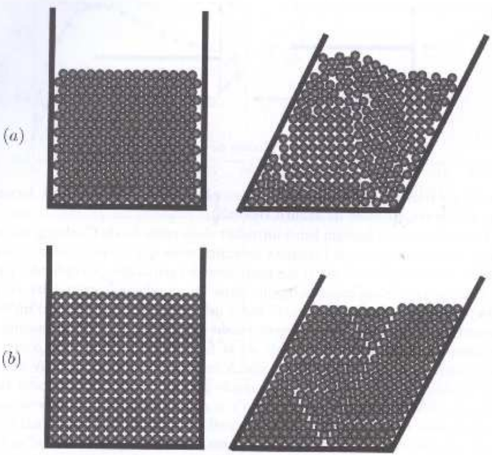


FIG. 4.9 – (a) Dilatance observée lorsqu'on cisaille un empilement bidimensionel triangulaire. (b) Contractance observée lorsqu'on cisaille un empilement carré (dessins inspirés d'expériences de Brown & Richards, 1970).

Figures from "Les milieux granulaires", B. Andreotti, Y. Forterre, O. Pouliquen

(a) The material dilates when deformed

(b) The material contracts when deformed

Relation between deformation and dilation/contraction !

We consider a wet granular material.
Figures from "Les milieux granulaires", B. Andreotti, Y. Forterre, O. Pouliquen

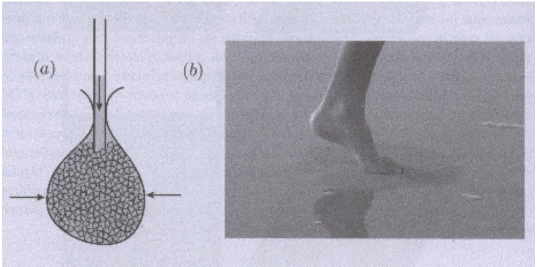


FIG. E4.1 – Exemple de phénomène induit par la dilatance des milieux granulaires. (a) Expérience de Reynolds. (b) Assèchement du sable sous le pied.

- (a), (b) When a dense packing of sand is deformed, the granular material dilates, and water is sucked
- When a loose packing of sand is deformed, the granular material contracts, and water is expelled (ex : landslide, quicksand "sables mouvants").

Great open problems :

- Write a geophysically and mathematically relevant rheology for compressible viscoplastic materials !
- Derive (even formally !) viscoplastic laws from a microscopic description

Basis of classical thin layer approximations :

- Saint Venant (shallow water) model derived from incompressible Euler equations above an almost flat topography
- Savage, Hutter (1989) take into account the bottom friction and main curvature,
- Submerged flows models proposed by Iverson, Denlinger (2001), Pitman, Le (2005), Pailha, Pouliquen (2009), Iverson, George (2014)

Difficulties :

- Complex rheology not taken into account in the above expansions, but only dilatancy effects.
- Physics of diphasic mixtures is involved. Dilatancy laws are not obvious to set.

Benefits :

- Replace viscoplatic laws by boundary friction
- Less space variables is better for simulation

In this talk :

- Use asymptotic expansions as far as possible without introducing heuristic rules.
- Keep the mathematical structure to avoid unphysical effects. In particular, keep an energy balance.

We consider a wet granular material. The features to take into account are

- buoyancy force
- interphase friction force (*drag*)
- solid (granular) pressure and pore (fluid) pressure, both non hydrostatic because of the relative interphase velocity
- Dilatancy : the granular material dilates if its volume fraction is larger than a critical value, or contracts in the opposite case. In the first case the granular friction increases (stiffening of the granular matrix). In the second case the granular friction diminishes (fluidisation). Law proposed by Roux and Radjai (1998).

Jackson's model can be written as follows.

- Two mass equations

$$\boxed{\begin{aligned}\partial_t(\rho_s\varphi) + \nabla\cdot(\rho_s\varphi v) = 0,\\ \partial_t(\rho_f(1-\varphi)) + \nabla\cdot(\rho_f(1-\varphi)u) = 0,\end{aligned}}$$

  with ρ_s, ρ_f the (constant) solid/fluid mass densities, φ the solid volume fraction, v the solid velocity, and u the fluid velocity.

- Two momentum equations

$$\boxed{\begin{aligned}\rho_s\varphi(\partial_t v + v\cdot\nabla v) = -\nabla\cdot T_s - \varphi\nabla p_{f_m} + f + \rho_s\varphi\mathbf{g},\\ \rho_f(1-\varphi)(\partial_t u + u\cdot\nabla u) = -\nabla\cdot T_{f_m} + \varphi\nabla p_{f_m} - f + \rho_f(1-\varphi)\mathbf{g},\end{aligned}}$$

  where $\mathbf{g}$ is gravity, f is the drag force

$$f = \tilde{\beta}(u-v), \qquad \tilde{\beta} = \frac{18\nu_f\rho_f\varphi}{d^2}$$

  with ν_f the fluid viscosity and d the diameter of the grains.

- T_s, T_{f_m} are the solid and fluid stress tensors, taken as

$$T_s = p_s\mathrm{I}d + \widetilde{T}_s, \qquad T_{f_m} = p_{f_m}\mathrm{I}d + \widetilde{T}_{f_m},$$

  with $\widetilde{T}_k$ small far from shearing boundary layers.

Unknowns in Jackson's model :

- 5 unknowns : φ, v (vector), u (vector), p_s, p_{f_m}
- 4 equations (2 vectors)

A constitutive equation is required to close the system : dilatancy law

The only non Newtonian effects that we take into account result from dilatancy.

Qualitative explanation of dilatancy effects

$\varphi < \varphi_c$ (loose)

Granular medium contracts when deformed

⬇

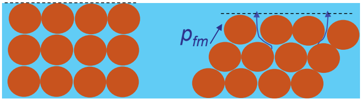
Water must be expelled

⬇

Increasing pore pressure

⬇

Liquefaction



$\varphi > \varphi_c$ (dense)

Granular medium dilates when deformed

⬇

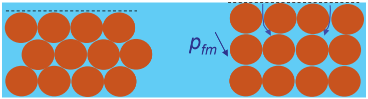
Water must be sucked

⬇

Decreasing pore pressure

⬇

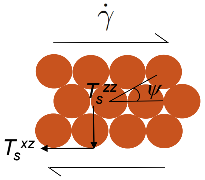
Stiffening of the granular matrix



Pailha and Pouliquen, 2009; Y. Forterre, Seville 2014

Modeling of dilatancy effects

- Compression/dilatation of the solid phase :



$$\nabla \cdot v = \dot{\gamma} \tan \psi$$ *Roux and Radjai, 1998*

ψ : dilatancy angle, $\dot{\gamma}$: shear strain rate

$$\tan \psi = K(\varphi - \varphi_c)$$ *Pailha and Pouliquen, 2009*

➡ $\boxed{\text{Closure equation : } \nabla \cdot v = K \dot{\gamma} (\varphi - \varphi_c)}$

$\varphi < \varphi_c$: compression $\qquad$ $\varphi > \varphi_c$: dilatation

- Impact of the dilatancy angle on the Coulomb friction force :

$$\boxed{T_s^{\mathbf{x}z} = -\tan(\delta + \psi)\,\mathrm{sign}(v)\, T_s^{zz}}$$

Energy balance

- With a slighly modified Roux-Radjai closure, Jackson's system has a relevant energy equation

$$\begin{aligned}
&\partial_t\left(\rho_s\varphi\frac{|v|^2}{2}+\rho_f(1-\varphi)\frac{|u|^2}{2}-(\mathbf{g}\cdot X)\big(\rho_s\varphi+\rho_f(1-\varphi)\big)+\rho_s\varphi e_c\right)\\
&+\nabla\cdot\left(\rho_s\varphi\frac{|v|^2}{2}v+\rho_f(1-\varphi)\frac{|u|^2}{2}u-(\mathbf{g}\cdot X)\big(\rho_s\varphi v+\rho_f(1-\varphi)u\big)\right.\\
&\qquad\left.+p_{f_m}\big(\varphi v+(1-\varphi)u\big)+\widetilde{T}_{f_m}u+T_s v+\rho_s\varphi e_c v\right)\\
&=(p_s-p_c)\nabla\cdot v+\widetilde{T}_s:\nabla v+\widetilde{T}_{f_m}:\nabla u+f\cdot(v-u).
\end{aligned}$$

  with X the space position, and the closure is taken as

$$\nabla\cdot v=K_p\dot\gamma(p_c(\varphi)-p_s)$$

  with $p_c(\varphi)$ the critical pressure, and $e_c(\varphi)$ the critical specific internal energy defined by $de_c/d\varphi=p_c/(\rho_s\varphi^2)$.
- The right-hand side of the energy balance equation is nonpositive
- The closure law can be interpreted as a compressible bulk viscoplastic rheology

$$p_s=p_c(\varphi)-\frac{\nabla\cdot v}{K_p\dot\gamma},$$

  with $\dot\gamma=||Dv||$, $Dv=(\nabla v+(\nabla v)^t)/2$.

Bouchut, Fernandez-Nieto, Mangeney, Narbona-Reina (2015).

- Two thin layers : one layer of solid/fluid mixture (volume fraction $0 < \varphi < 1$) treated via Jackson's model, and one layer of pure fluid above

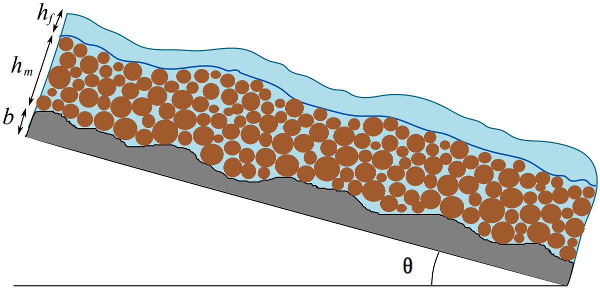


FIG.: Two-layer configuration

- The fluid can enter or get out of the mixture when the granular medium dilates or contracts
- In contrast with all previously proposed models, we have $h_f > 0$, and two unknown velocities (one for the solid and one for the fluid) instead of one.
- Suitable boundary conditions have to be set.

- At the bottom we put non penetration conditions

$$u \cdot n = 0, \qquad v \cdot n = 0,$$

a solid Coulomb friction law

$$(T_s n)_\tau = -\tan\delta_{\mathrm{eff}} \frac{v}{|v|}(T_s n)\cdot n,$$

and a Navier friction condition for the fluid phase

$$(T_{f_m} n)_\tau = -k_b u.$$

- At the free surface we assume no tension for the fluid

$$T_f N_X = 0,$$

together with the kinematic condition

$$N_t + u_f \cdot N_X = 0,$$

where $N = (N_t, N_X)$ is a time-space normal to the free surface.

- At the interface, we have the kinematic condition for the solid phase

$$\tilde{N}_t + v \cdot \tilde{N}_X = 0,$$

where we denote by $\tilde{N} = (\tilde{N}_t, \tilde{N}_X)$ a time-space upward normal to the interface.
The total fluid mass is conserved,

$$\tilde{N}_t + u_f \cdot \tilde{N}_X = (1 - \varphi^*)(\tilde{N}_t + u \cdot \tilde{N}_X) \equiv \mathcal{V}_f$$

where φ^* is the value of the solid volume fraction at the interface (the limit is taken from the mixture side). The term $\mathcal{V}_f$ defines the fluid mass that is transferred from the mixture to the fluid-only layer.
The conservation of the total momentum is written

$$\rho_f \mathcal{V}_f (u - u_f) + (T_s + T_{f_m})\tilde{N}_X = T_f \tilde{N}_X.$$

The energy balance through the interface yields the stress transfer condition

$$T_s \tilde{N}_X = \left(\frac{\rho_f}{2} \left((u - u_f) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|} \right)^2 \right. \\ \left. + \left((T_{f_m} \tilde{N}_X) \cdot \frac{\tilde{N}_X}{|\tilde{N}_X|^2} - p_{f_m} \right) \frac{\varphi^*}{1 - \varphi^*} \right) \tilde{N}_X.$$

These conditions are completed by a Navier fluid friction condition

$$\left(\frac{T_{f_m} + T_f}{2} \tilde{N}_X \right)_\tau = -k_i (u_f - u)_\tau.$$

Fluid pore pressure

in thin layer depth-averaged models

- From vertical momentum conservation

* Fluid: $$\varepsilon\rho_f(1-\varphi)\cancel{(\partial_t u^z + u^{\mathbf{x}}\cdot\nabla_{\mathbf{x}}u^z + u^z\partial_z u^z)} = -(1-\varphi)\partial_z p_{f_m} - g\cos\theta\rho_f(1-\varphi) - \beta(u^z - v^z) - \cancel{\varepsilon\nabla_{\mathbf{x}}\cdot T_{f_m}^{\mathbf{x}z}}$$

* Solid: $$\varepsilon\rho_s\varphi\cancel{(\partial_t v^z + v^{\mathbf{x}}\cdot\nabla_{\mathbf{x}}v^z + v^z\partial_z v^z)} = -\partial_z p_s - \varphi\partial_z p_{f_m} - g\cos\theta\rho_s\varphi + \beta(u^z - v^z) - \cancel{\varepsilon\nabla_{\mathbf{x}}\cdot T_s^{\mathbf{x}z}}$$

➡ $$p_{f_m} = \rho_f g\cos\theta(b + h_m + h_f - z) + \widetilde{p^e_{f_m}}$$

➡ $$p_s = (\rho_s - \rho_f)g\cos\theta\bar{\varphi}(b + h_m - z) - \widetilde{p^e_{f_m}}$$

using the closure equation $\nabla\cdot v = K\dot{\gamma}(\varphi - \varphi_c)$

⬇

- Non-hydrostatic fluid pressure

$$\widetilde{p^e_{f_m}} = \frac{\bar{\beta}}{1-\bar{\varphi}}h_m(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}})\cdot\nabla_{\mathbf{x}}b - \frac{\bar{\beta}}{1-\bar{\varphi}}\frac{h_m^2}{2}\left(\nabla_{\mathbf{x}}\cdot(\overline{u^{\mathbf{x}}} - \overline{v^{\mathbf{x}}}) + \frac{K\bar{\dot{\gamma}}(\bar{\varphi} - \varphi_c)}{1-\bar{\varphi}}\right)$$

- *Bouchut, Fernandez-Nieto, Mangeney, Narbona-Reina's model*

$$\begin{aligned}\rho_s\bar{\varphi}(\partial_t v + v\nabla_{\mathbf{x}} v) &= -(\rho_s-\rho_f)g\cos\theta\frac{h_m}{2}\nabla_{\mathbf{x}}\bar{\varphi} + (1-\bar{\varphi})\nabla_{\mathbf{x}}\widetilde{p^e_{f_m}}\\ &\quad -g\cos\theta\bar{\varphi}\big(\rho_s\nabla_{\mathbf{x}}(b+h_m)\big)\\ &\quad -\mathrm{sign}(v)(\tan\delta + K(\bar{\varphi}-\varphi_c))\frac{1}{h_m}\Big((\rho_s-\rho_f)g\cos\theta\bar{\varphi}h_m - \widetilde{p^e_{f_m}}\Big)\\ &\quad +\bar{\beta}(u-v) - \rho_s g\sin\theta\bar{\varphi}(1,0)^t.\end{aligned}$$

$$\begin{aligned}\rho_f(1-\bar{\varphi})\big(\partial_t u + u\nabla_{\mathbf{x}} u\big) &= -\rho_f g\cos\theta(1-\bar{\varphi})\nabla_{\mathbf{x}}(b+h_m) - (1-\bar{\varphi})\nabla_{\mathbf{x}}\widetilde{p^e_{f_m}}\\ &\quad -\bar{\beta}(u-v) - (1-\bar{\varphi})\rho_f g\sin\theta(1,0)^t\\ &\quad -\frac{1}{h_m}\left(\left(\frac{\rho_f\mathcal{V}_f}{2}-\alpha\right)(\overline{u_f^{\mathbf{x}}}-u)\right)\end{aligned}$$

$$\widetilde{p^e_{f_m}} = \frac{\bar{\beta}}{1-\bar{\varphi}}h_m(u-v)\cdot\nabla_{\mathbf{x}} b - \frac{\bar{\beta}}{1-\bar{\varphi}}\frac{h_m^2}{2}\left(\nabla_{\mathbf{x}}\cdot(u-v) + \frac{K\bar{\dot{\gamma}}(\bar{\varphi}-\varphi_c)}{1-\bar{\varphi}}\right)$$

- *Iverson and George's model*

Isotropy of normal stress

$$\begin{aligned}\bar{\rho}_m\big(\partial_t v_m + v_m\nabla_{\mathbf{x}} v_m\big) &= -\bar{\rho}g\cos\theta\nabla_{\mathbf{x}}(b+h_m) + \cancel{(k-1)\nabla_{\mathbf{x}}\widetilde{p^e_{f_m}}}\\ &\quad -\mathrm{sign}(v_m)(\tan(\delta+\psi))\frac{1}{h_m}\Big((\rho_s-\rho_f)g\cos\theta\bar{\varphi}h_m - \widetilde{p^e_{f_m}}\Big)\end{aligned}$$

$$\partial_t\widetilde{p^e_{f_m}} - \nabla\cdot\left(\frac{\kappa}{\alpha\mu}\nabla\widetilde{p^e_{f_m}}\right) = -\frac{\dot{\gamma}\tan\psi}{\alpha} + \partial_t(\sigma - \rho_f g(h-z)\cos\theta)$$

Our thin-layer averaged model has three scalar equations on h_m, h_f, φ instead of two :

$$\boxed{\begin{aligned} &\partial_t(\bar\varphi h_m) + \nabla_{\mathbf{x}}\cdot(\bar\varphi h_m \overline{v^{\mathbf{x}}}) = 0,\\ &\partial_t\big((1-\bar\varphi)h_m + h_f\big) + \nabla_{\mathbf{x}}\cdot\big((1-\bar\varphi)h_m\overline{u^{\mathbf{x}}} + h_f\overline{u_f^{\mathbf{x}}}\big) = 0,\\ &\partial_t\bar\varphi + \overline{v^{\mathbf{x}}}\cdot\nabla_{\mathbf{x}}\bar\varphi. = -\bar\varphi\bar\Phi. \end{aligned}}$$

with

$$\bar\Phi = K\bar{\dot\gamma}(\bar\varphi - \bar\varphi_c).$$

It satisfies

- conservation of granular and fluid mass
- conservation of total momentum
- decrease of energy
- dilatancy with excess pore pressure, and fluidisation/stiffening of the granular material
- convergence to hydrostatic equilibrium in case of no external forces
- diffusion term on the relative velocity similar to the George-Iverson model

Numerical simulations in progress !

Open problem : include viscoplastic rheology with yield stress

Conclusion

The modeling of wet granular materials leads to several very difficult issues

- The used (incompressible) viscoplastic laws are not justified from microscopic laws
- Need to write a geophysically and mathematically relevant rheology for compressible viscoplastic materials with yield stress and dilatancy
- Need to derive associated thin-layer models

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We're nothing, and nothing will help us
Maybe we're lying, then you'd better not stay
But we could be safer, just for one day

D. Bowie "Heroes"