Reconstruction of an immersed obstacle from boundary measurements

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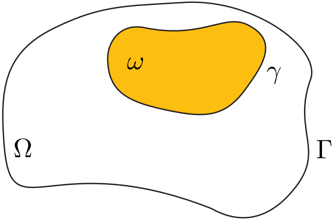
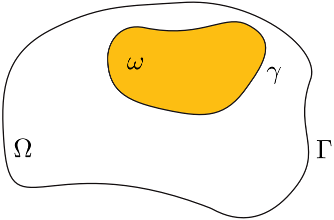
The problem in a glance



Settings

- ω is the obstacle immersed in a perfect fluid.
- Ω is the domain of the fluid.
- φ is the harmonic potential of the fluid in Ω and then $\nabla\varphi$ is the fluid velocity field.
- The *slip* boundary condition on γ reads: $\partial_n \varphi = 0$.

The problem in a glance



Problem statement

Knowing the Neumann-to-Dirichlet map:

$$\partial_n \varphi \in H^{-\frac{1}{2}}(\Gamma) \mapsto \varphi \in H^{\frac{1}{2}}(\Gamma),$$

how to reconstruct the obstacle ω?

The problem in a glance

Restatement in terms of the stream function

The stream function ψ ($\varphi + i\psi$ is holomorphic in Ω) sastifies:

$$\begin{aligned} -\Delta\psi &= 0 \quad \text{in } \Omega \\ \psi &= c \quad \text{on } \gamma \\ \psi &= f \quad \text{on } \Gamma \end{aligned}$$

where the constant $c \in \mathbb{R}$ is such that:

$$\int_\gamma \partial_n \psi = 0.$$

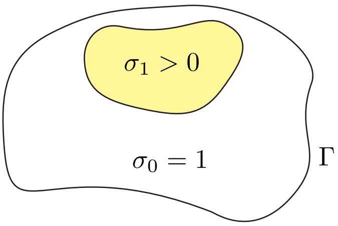
Problem statement

Knowing the Dirichlet-to-Neumann (DtN) map:

$$f \in H^{\frac{1}{2}}(\Gamma) \mapsto \partial_n \psi \in H^{-\frac{1}{2}}(\Gamma),$$

how to reconstruct the obstacle ω?

Calderón's conductivity problem



$$\begin{aligned} -\mathrm{div}(\sigma \nabla u) &= 0 \quad \text{in } \Omega_0 \subset \mathbb{R}^2, \\ u &= f \quad \text{on } \Gamma. \end{aligned}$$

Problem statement

Knowing the Dirichlet-to-Neumann (DtN) map:

$$f \in H^{\frac{1}{2}}(\Gamma) \mapsto \partial_n u \in H^{-\frac{1}{2}}(\Gamma),$$

how to reconstruct the piecewise constant conductivity σ?

Calderón's conductivity problem

We recover the initial problem by letting $\sigma_1 \to +\infty$ (Highly conducting inclusion):

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega, \\ u &= f \quad \text{on } \Gamma, \\ u &= c \quad \text{on } \gamma, \end{aligned}$$

where c is the constant such that: $\displaystyle\int_\gamma \partial_n u = 0.$

Bibliographical comments

Identifiability

Only one measurement required (straightforward) see e.g. Alessandrini & Rondi, 2001, Kress, 2004.

Stability

Logarithmic stability is best possible; see e.g. Alessandrini (IP, 2007) or Uhlmann (IP, 2009) and references therein.

Bibliographical comments

Reconstruction

Iterative methods

- Optimization methods: Borcea, Dobson, Hanke, Santosa,...
- Quasi-reversibility + Level Sets: Bourgeois *et al.*...
- Conformal mapping method: Akduman, Haddar, Kress,...

Non iterative methods

- Nachman's direct reconstruction method: Siltanen *et al.*
- Indicator functions:
  - **Enclosure/probe method**: Ikehata *et al.*, Nakamura, Potthast,...
  - **LSM/Factorization methods**: Brühl & Hanke, Cakoni, Colton, Kirsch, Haddar, Kress,...
- Asymptotic methods & Generalized Polya-Szegö Tensors: Ammari *et al.*, Vogelius *et al.*, Kang*et al.*..

Background on single layer potential (1/3)

For every given density $\widehat{q}$ on $\Gamma = \partial\Omega$, the single layer potential is defined by

$$\mathscr{S}_\Gamma \widehat{q}(x) = \int_\Gamma G(x-y)\widehat{q}(y)\,\mathrm{d}\sigma_y, \quad x \notin \Gamma,$$

where

$$G(x) = -\frac{1}{2\pi}\log|x|$$

denotes the fundamental solution of $-\Delta$ in $\mathbb{R}^2$.

Background on single layer potential (2/3)

- $\mathscr{S}_\Gamma \widehat{q}$ is harmonic in $\mathbb{R}^2 \setminus \Gamma$.

- $\mathscr{S}_\Gamma \widehat{q}$ satisfies the following jump conditions:

$$[\mathscr{S}_\Gamma \widehat{q}]_{|\Gamma} = 0, \qquad [\partial_n (\mathscr{S}_\Gamma \widehat{q})]_{|\Gamma} = \widehat{q}.$$

- The trace of the single layer potential, denoted by S_Γ defines a bounded operator from $H^{-\frac{1}{2}}(\Gamma)$ to $H^{\frac{1}{2}}(\Gamma)$.
  We denote:

$$q = \mathsf{S}_\Gamma \widehat{q}.$$

- If $\mathrm{Cap}(\Gamma) \neq 1$, S_Γ is an isometry provided $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$ are endowed with the norms:

$$\|\widehat{q}\|^2_{-\frac{1}{2}} = \langle \widehat{q}, \mathsf{S}_\Gamma \widehat{q} \rangle = \langle \mathsf{S}_\Gamma^{-1} q, q \rangle = \|q\|^2_{\frac{1}{2}}.$$

Background on single layer potential (3/3)

- The equilibrium density $\widehat{\mathsf{e}}_\Gamma \in H^{-\frac{1}{2}}(\Gamma)$ is the unique density such that:

$$\mathsf{S}_\Gamma \widehat{\mathsf{e}}_\Gamma \text{ is constant on } \Gamma \text{ and } \langle \widehat{\mathsf{e}}_\Gamma, 1 \rangle = 1.$$

- The operator S_Γ defines an isometry between the spaces

$$\widehat{H}(\Gamma) := \left\{ \widehat{q} \in H^{-\frac{1}{2}}(\Gamma) \, : \, \langle \widehat{q}, 1 \rangle = 0 \right\},$$
$$H(\Gamma) := \left\{ q \in H^{\frac{1}{2}}(\Gamma) \, : \, \langle \widehat{\mathsf{e}}_\Gamma, q \rangle = 0 \right\}.$$

  We introduce the projections:

$$\Pi_\Gamma : H^{\frac{1}{2}}(\Gamma) \to H(\Gamma) \text{ and } \widehat{\Pi}_\Gamma : H^{-\frac{1}{2}}(\Gamma) \to \widehat{H}(\Gamma).$$

- The following equivalence holds for $q \in H^{\frac{1}{2}}(\Gamma)$:

$$q \in H(\Gamma) \Leftrightarrow \int_{\mathbb{R}^2} |\nabla(\mathscr{S}_\Gamma \widehat{q})|^2 < +\infty.$$

  In this case:

$$\|\widehat{q}\|^2_{-\frac{1}{2}} = \|q\|^2_{\frac{1}{2}} = \int_{\mathbb{R}^2} |\nabla(\mathscr{S}_\Gamma \widehat{q})|^2.$$

Back to the DtN

- The DtN operator is by assumption valued in $\widehat{H}(\Gamma)$ since:

$$\int_\Gamma \partial_n u = -\int_\gamma \partial_n u = 0.$$

- We will consider its restriction to $H(\Gamma)$ and we denote:

$$\Lambda_\gamma : f \in H(\Gamma) \longmapsto \partial_n u\big|_\Gamma \in \widehat{H}(\Gamma).$$

- When $\omega = \varnothing$ (there is no obstacle), we denote the DtN by Λ_0.
- Most of time, we will consider the operator:

$$\mathsf{R} := \mathsf{S}_\Gamma(\Lambda_\gamma - \Lambda_0) : H(\Gamma) \to H(\Gamma).$$

The measurements

Let us define the harmonic polynomials:

$$\mathcal{Z}_\Gamma^m := \Pi_\Gamma z^m \quad \text{and} \quad \overline{\mathcal{Z}}_\Gamma^m := \Pi_\Gamma \bar{z}^m, \quad (m \geqslant 1),$$

and recall that $\mathsf{R} := \mathsf{S}_\Gamma(\Lambda_\gamma - \Lambda_0) : H(\Gamma) \to H(\Gamma)$.

Proposition

The operator $\mathrm{Id} + \mathsf{R} : H(\Gamma) \to H(\Gamma)$ is invertible and we can define the complex sequences:

$$\mu_m = \frac{1}{2} \langle \mathsf{R}(\mathrm{Id} + \mathsf{R})^{-1} \overline{\mathcal{Z}}_\Gamma^1, \mathcal{Z}_\Gamma^m \rangle_{\frac{1}{2},\Gamma},$$

$$\nu_m = \frac{1}{2} \langle \mathsf{R}(\mathrm{Id} + \mathsf{R})^{-1} \mathcal{Z}_\Gamma^1, \mathcal{Z}_\Gamma^m \rangle_{\frac{1}{2},\Gamma}, \qquad (m \geqslant 1).$$

- R is known, so are the sequences $(\mu_m)_{m \geqslant 1}$ and $(\nu_m)_{m \geqslant 1}$.
- The numbers μ_m and ν_m are closely related with the Generalized Pólya-Szegö Tensors appearing in the asymptotic expansion of the DtN for small inclusions (see Ammari *et al.*).

The conformal mapping

The boundary γ can be described through the conformal mapping that maps the exterior of the unit disk onto the exterior of ω ($a_1 > 0$):

$$\phi_\gamma : z \mapsto a_1 z + a_0 + \sum_{m \geqslant 1} a_{-m} z^{-m}.$$

In this description:

- $a_1 > 0$ is the (logarithmic) capacity of γ.
- a_0 is the conformal center.

The problem of reconstructing the cavity ω is equivalent to the problem of recovering the complex sequence $(a_k)_{k \leqslant 1}$.

Explicit reconstruction formula

Theorem

We have $\mu_1 > 0$ and the coefficients a_k can be computed by means of the following formulae:

$$a_1 = \left(\frac{\mu_1}{2\pi}\right)^{\frac{1}{2}} \qquad a_0 = \frac{\mu_2}{2\mu_1}$$

$$a_{-m} = \mu_1^{-\frac{m}{2}} \sum_{\alpha \in \mathsf{A}_m} C_\alpha \left(\frac{\mu_2}{\mu_1}\right)^{\alpha_0} \nu_1^{\alpha_1} \nu_2^{\alpha_2} \dots \nu_m^{\alpha_m}, \quad m \geqslant 1,$$

where

$$\mathsf{A}_m := \{\alpha \in \mathbb{N}^{m+1} \,:\, \alpha_0 + 2\alpha_1 + 3\alpha_2 + \ldots + (m+1)\alpha_m = (m+1)\}$$

and

$$C_\alpha := \frac{(-1)^{|\alpha|+1}}{2^{\alpha_0} m} \frac{(2\pi)^{\frac{m}{2} - (\alpha_1 + \cdots + \alpha_m)}}{1^{\alpha_1} 2^{\alpha_2} \ldots m^{\alpha_m}}.$$

Algorithm

1. The space $H(\Gamma)$ is approximated by the finite dimensional space spanned by the family $\{\mathcal{Z}_\Gamma^m, \overline{\mathcal{Z}}_\Gamma^m,\, m = 1, \dots, M\}$ where

$$\mathcal{Z}_\Gamma^m := \Pi_\Gamma z^m \quad \text{and} \quad \overline{\mathcal{Z}}_\Gamma^m := \Pi_\Gamma \bar{z}^m, \quad (m \geqslant 1).$$

2. We compute the $2M \times 2M$ matrix $\mathbf{Q}_M$ whose entries are:

$$\langle \overline{\mathcal{Z}}_\Gamma^m, \mathcal{Z}_\Gamma^{m'} \rangle_{\frac{1}{2},\Gamma} \text{ and } \langle \mathcal{Z}_\Gamma^m, \mathcal{Z}_\Gamma^{m'} \rangle_{\frac{1}{2},\Gamma} \quad 1 \leqslant m, m' \leqslant M.$$

3. We compute the $2M \times 2M$ matrix $\mathbf{R}_M$ whose entries are:

$$\langle \overline{\mathcal{Z}}_\Gamma^m, \mathsf{R}\mathcal{Z}_\Gamma^{m'} \rangle_{\frac{1}{2},\Gamma} \text{ and } \langle \mathcal{Z}_\Gamma^m, \mathsf{R}\mathcal{Z}_\Gamma^{m'} \rangle_{\frac{1}{2},\Gamma} \quad 1 \leqslant m, m' \leqslant M.$$

4. An approximation of the vector $(\mu_1, \nu_1, \dots, \mu_M, \nu_M)$ is given by the first raw of the matrix product:

$$\mathbf{Q}_M(\mathbf{Q}_M + \mathbf{R}_M)^{-1}\mathbf{R}_M.$$

5. We use the formulae of the Theorem to compute the coefficients $a_1, a_0, a_{-1}, \dots, a_{-M}$.

Numerical results

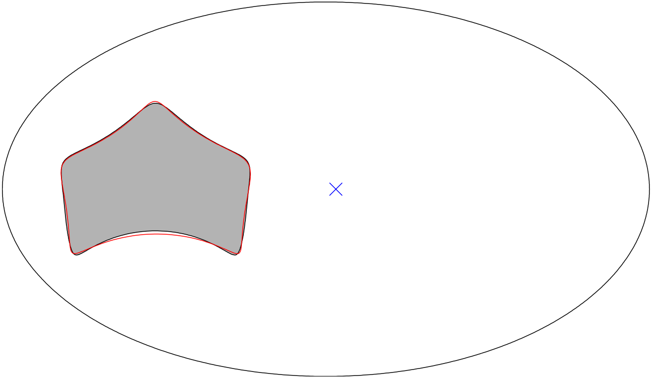


Figure : Examples of reconstructions with 8 (complex) coefficients.

Computations are made with the `Matlab` Laplace boundary integral equation solver IES (B. Pinçon and A. M.).

Main ingredients of the proof

Step 1 Getting rid of the outer boundary Γ

The operator $\mathsf{K} := (\mathrm{Id} + \mathsf{R})^{-1}\mathsf{R}$ satisfies, for every $f, g \in H(\Gamma)$:

$$\langle \mathsf{K} f, g \rangle_{\frac{1}{2},\Gamma} = \langle \Pi_\gamma(\mathscr{S}_\Gamma f), \Pi_\gamma(\mathscr{S}_\Gamma g) \rangle_{\frac{1}{2},\gamma}.$$

Step 2 A suitable choice of test functions

Specifying f and g to be the harmonic polynomials we get:

$$\langle \mathsf{K} \mathcal{Z}^m_\Gamma, \mathcal{Z}^{m'}_\Gamma \rangle_{\frac{1}{2},\Gamma} = \langle \mathcal{Z}^m_\gamma, \mathcal{Z}^{m'}_\gamma \rangle_{\frac{1}{2},\gamma}$$

$$\langle \mathsf{K} \mathcal{Z}^m_\Gamma, \overline{\mathcal{Z}}^{m'}_\Gamma \rangle_{\frac{1}{2},\Gamma} = \langle \mathcal{Z}^m_\gamma, \overline{\mathcal{Z}}^{m'}_\gamma \rangle_{\frac{1}{2},\gamma}.$$

Step 3 Complex analysis tools

Using the conformal mapping we can compute:

$$\langle \mathcal{Z}^m_\gamma, \mathcal{Z}^{m'}_\gamma \rangle_{\frac{1}{2},\gamma} \text{ and } \langle \mathcal{Z}^m_\gamma, \overline{\mathcal{Z}}^{m'}_\gamma \rangle_{\frac{1}{2},\gamma} = \text{function of } a_k \ (k \leqslant 1).$$

The identity above can be inverted:

$$a_k = \text{function of } \underbrace{\langle \mathcal{Z}^m_\gamma, \mathcal{Z}^1_\gamma \rangle_{\frac{1}{2},\gamma}}_{\nu_m} \text{ and } \underbrace{\langle \mathcal{Z}^m_\gamma, \overline{\mathcal{Z}}^1_\gamma \rangle_{\frac{1}{2},\gamma}}_{\mu_m}, \ (m \geqslant 1).$$

Boundary integral formulation (step 0)

Theorem

For all $f \in H^{\frac{1}{2}}(\Gamma)$, there exists a unique $(u, c) \in H^1(\Omega) \times \mathbb{R}$ such that:

$$\begin{aligned} -\Delta u &= 0 \quad && \text{in } \Omega, \\ u &= f && \text{on } \Gamma, \\ u &= c && \text{on } \gamma, \\ \int_\gamma \partial_n u &= 0. \end{aligned}$$

Moreover, u admits a single layer representation

$$u = \mathscr{S}_\Gamma \widehat{q} + \mathscr{S}_\gamma \widehat{p},$$

where $\widehat{q} \in H^{-\frac{1}{2}}(\Gamma)$ and $\widehat{p} \in H^{-\frac{1}{2}}(\gamma)$ satisfy:

$$\begin{aligned} q + (\mathscr{S}_\gamma \widehat{p})_\Gamma &= f \qquad && (\Gamma), \\ (\mathscr{S}_\Gamma \widehat{q})_\gamma + p &= c && (\gamma). \end{aligned}$$

Factorization of the DtN (step 1)

Lemma

If $f \in H(\Gamma)$ then $\widehat{p} \in \widehat{H}(\gamma)$ and $\widehat{q} \in \widehat{H}(\Gamma)$.

Applying the projections Π_Γ and Π_γ to the system:

$$q + (\mathscr{S}_\gamma \widehat{p})_\Gamma = f \qquad (\Gamma),$$

$$(\mathscr{S}_\Gamma \widehat{q})_\gamma + p = c \qquad (\gamma),$$

we get:

$$\begin{cases} q + \mathsf{K}^+ p = f & (\Gamma), \\ \mathsf{K}^- q + p = 0 & (\gamma), \end{cases}$$

where the operators K^+ and K^- are defined by:

$$\mathsf{K}^+ p := \Pi_\Gamma (\mathscr{S}_\gamma \widehat{p})_{|\Gamma} \qquad \mathsf{K}^- q := \Pi_\gamma (\mathscr{S}_\Gamma \widehat{q})_{|\gamma}.$$

Factorization of the DtN (step 1)

Proposition

The operators $\mathsf{K}^{\pm}$ enjoy the following properties:

- K^{+} (respectively K^{-}) is compact from $H(\gamma)$ onto $H(\Gamma)$ (respectively from $H(\Gamma)$ onto $H(\gamma)$).
- For all $p, q \in H(\gamma) \times H(\Gamma)$:

$$\langle \mathsf{K}^{+} p, q \rangle_{\frac{1}{2},\Gamma} = \langle p, \mathsf{K}^{-} q \rangle_{\frac{1}{2},\gamma}.$$

- $\mathsf{K}^{+} : H(\gamma) \to H(\Gamma)$ and $\mathsf{K}^{-} : H(\Gamma) \to H(\gamma)$ are contraction operators:

$$\|\mathsf{K}^{\pm}\| < 1.$$

Factorization of the DtN (step 1)

Set now

$$\mathsf{K} := \mathsf{K}^{+}\mathsf{K}^{-} : H(\Gamma) \to H(\Gamma)$$

and recall that:

$$\mathsf{R} := \mathsf{S}_{\Gamma}(\Lambda_{\gamma} - \Lambda_{0}) : H(\Gamma) \to H(\Gamma).$$

Theorem (Factorization)

The following identities hold true:

$$\mathsf{R} = (\mathrm{Id} - \mathsf{K})^{-1}\mathsf{K},$$

or equivalently

$$\mathsf{K} = (\mathrm{Id} + \mathsf{R})^{-1}\mathsf{R}.$$

In other words, the knowledge of Λ_{γ} (and Λ_{0}) entirely determines the operator K .

Factorization of the DtN (steps 1-2)

The identity

$$(\mathrm{Id} + \mathsf{R})^{-1}\mathsf{R} = \mathsf{K}^{+}\mathsf{K}^{-}$$

reads equivalently for $f, g \in H(\Gamma)$:

$$\langle (\mathrm{Id} + \mathsf{R})^{-1}\mathsf{R} f, g\rangle_{\frac{1}{2},\Gamma} = \langle \mathsf{K}^{+}\mathsf{K}^{-} f, g\rangle_{\frac{1}{2},\Gamma} = \langle \mathsf{K}^{-} f, \mathsf{K}^{-} g\rangle_{\frac{1}{2},\gamma},$$

and by definition:

$$\langle \mathsf{K}^{-} f, \mathsf{K}^{-} g\rangle_{\frac{1}{2},\gamma} = \langle \Pi_{\gamma}(\mathscr{S}_{\Gamma} f), \Pi_{\gamma}(\mathscr{S}_{\Gamma} g)\rangle_{\frac{1}{2},\gamma}.$$

In particular, for the harmonic polynomials, we obtain:

$$\langle \mathsf{K} \mathcal{Z}^{m}_{\Gamma}, \mathcal{Z}^{m'}_{\Gamma}\rangle_{\frac{1}{2},\Gamma} = \langle \mathsf{K}^{-} \mathcal{Z}^{m}_{\Gamma}, \mathsf{K}^{-} \mathcal{Z}^{m'}_{\Gamma}\rangle_{\frac{1}{2},\gamma} = \langle \mathcal{Z}^{m}_{\gamma}, \mathcal{Z}^{m'}_{\gamma}\rangle_{\frac{1}{2},\gamma}.$$

Complex analysis (step 3)

The next (and last) step is to relate the data $\langle \mathcal{Z}^m_\gamma, \mathcal{Z}^{m'}_\gamma \rangle_{\frac{1}{2},\gamma}$ to the unknown geometry.
By definition:

$$\langle \mathcal{Z}^m_\gamma, \mathcal{Z}^{m'}_\gamma \rangle_{\frac{1}{2},\gamma} = \int_\gamma \widehat{\mathcal{Z}}^m_\gamma \mathcal{Z}^{m'}_\gamma,$$

where

$$\widehat{\mathcal{Z}}^m_\gamma = \left[\partial_n(\mathscr{S}_\gamma \mathcal{Z}^m_\gamma)\right]\Big|_\gamma .$$

The exterior Dirichlet problem:

$$\begin{aligned} -\Delta u &= 0 && \text{in } \mathbb{R}^2 \setminus \overline{\omega} \\ u &= \mathcal{Z}^m_\gamma && \text{on } \gamma \\ |u(x)| &= O(|x|^{-1}) && \text{as } |x| \to +\infty, \end{aligned}$$

can be explicitly solved by means of the conformal mapping that maps the exterior of the unit disk onto the exterior of ω ($a_1 > 0$):

$$\phi_\gamma : z \mapsto a_1 z + a_0 + \sum_{m \geqslant 1} a_{-m} z^{-m}.$$

Complex analysis (step 3)

In particular, we can prove:

Lemma

For every $m \geqslant 1$:

$$\mu_m := \frac{1}{2}\langle \mathcal{Z}_\gamma^m, \overline{\mathcal{Z}}_\gamma^1 \rangle_{\frac{1}{2},\gamma} = a_1 \int_{-\pi}^{\pi} e^{-it} \phi_\gamma^m(e^{it}) \mathrm{d}t,$$

and

$$\nu_m := \frac{1}{2}\langle \mathcal{Z}_\gamma^m, \mathcal{Z}_\gamma^1 \rangle_{\frac{1}{2},\gamma} = a_1 \int_{-\pi}^{\pi} e^{it} \phi_\gamma^m(e^{it}) \mathrm{d}t$$
$$= 2\pi a_1 \sum_{|\alpha|=-1} a_{\alpha_1} \dots a_{\alpha_m}.$$

These equalities can be inverted and provide the reconstruction formulae of the Theorem.

Comments

- As a Corollary of the main Theorem, we deduce that the coefficients a_k depend smoothly on the DtN map R.

- We use the family $\{\mathcal{Z}_{\gamma}^{m}, \overline{\mathcal{Z}}_{\gamma}^{m}\ m \geqslant 1\}$ (harmonic polynomials) as test functions. Other choices are possible as e.g.

$$\mathcal{Z}_{\gamma}^{m}(\cdot - r) \text{ with } r \in \mathbb{C}.$$

  This choice impacts the quality of the reconstruction.

- The factorization result generalizes to 3D and/or multiple obstacles.

Further numerical tests: Influence of the shift r

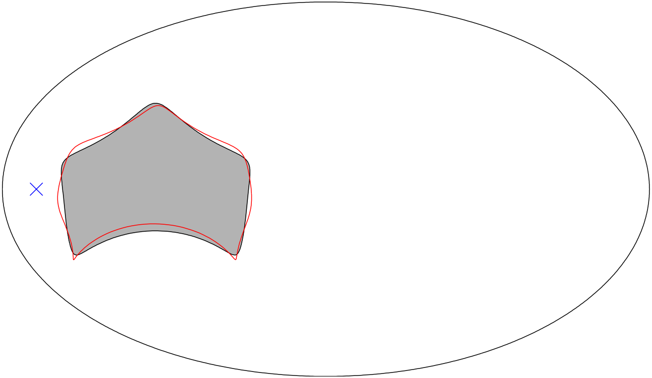
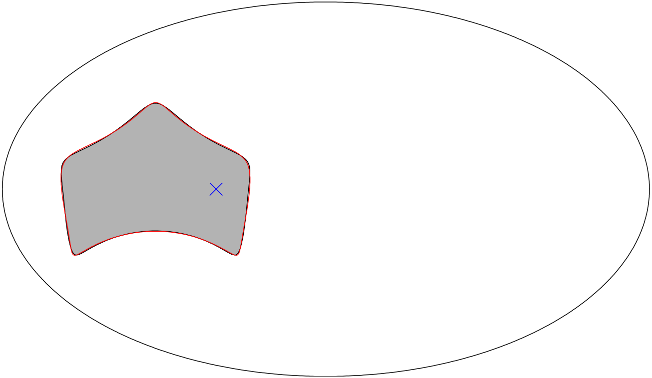


Figure : Examples of reconstructions with 8 (complex) coefficients.
Computations are made with the Matlab Laplace boundary integral equation solver IES (B. Pinçon and A. M.).

Further numerical tests: Influence of the shift r



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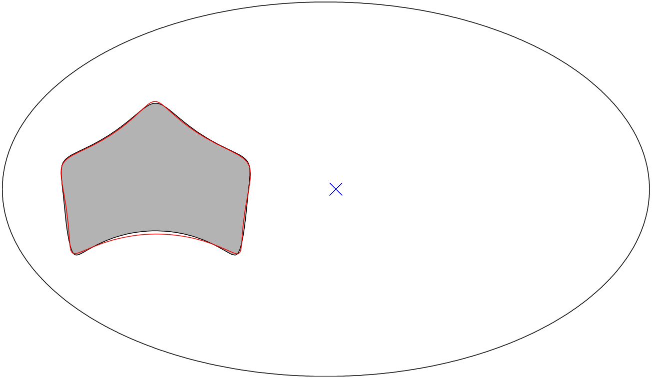

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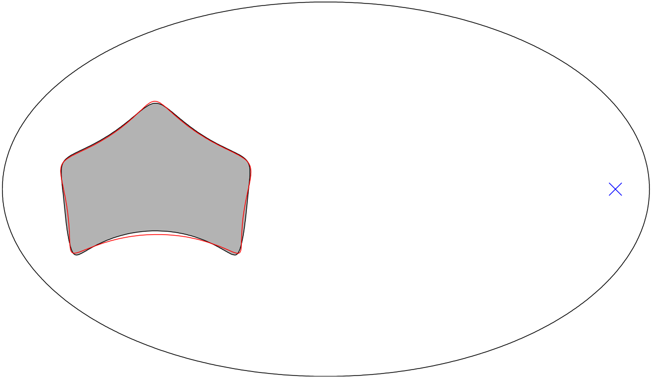

Figure : Examples of reconstructions with 8 (complex) coefficients.

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Further numerical tests: Noisy data

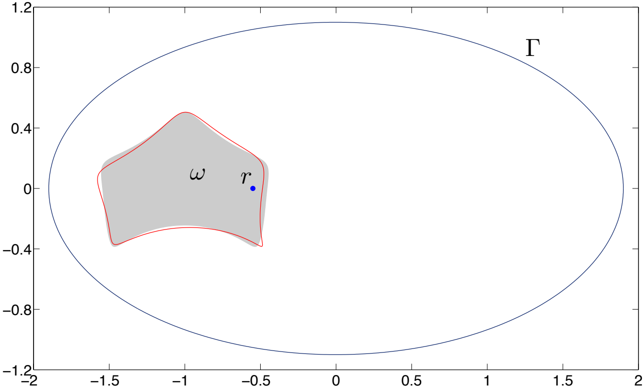
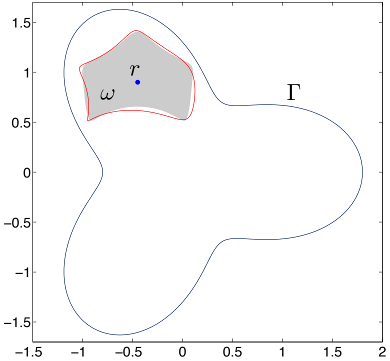
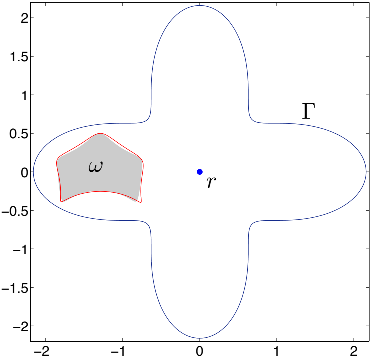


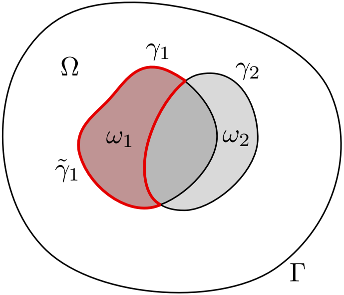
Figure : Reconstruction using $a_1, \ldots, a_{-4}$ with 5% of noise.

Further numerical tests: Influence of the outer boundary





Identifiability



Straightforward arguments requiring only one (non-constant) measurement:

- Assume that two cavities give the same measurement.
- Define $\psi = \psi_1 - \psi_2$.
- $\Delta\psi = 0$ and $\partial_n \psi = \psi = 0$ on Γ, hence $\psi = 0$.
- $c_1 = c_2 = c$ and $\psi_2 = c$ on $\tilde{\gamma}_1$. Then $\psi_2 = c$ in Ω.

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